Hindawi
Advances in Mathematical Physics
Volume 2019, Article ID 5658309, 7 pages
https://doi.org/10.1155/2019/5658309

Research Article

On the Coupling of the Homotopy Perturbation Method and New Integral Transform for Solving Systems of Partial Differential Equations

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Received 6 May 2019; Accepted 3 September 2019; Published 22 December 2019

Academic Editor: Qin Zhou

In the current work, a combination between a new integral transform and the homotopy perturbation method is presented. This combination allows to obtain analytic and numerical solutions for linear and nonlinear systems of partial differential equations.

1. Introduction

We know that the HPM, proposed first by He [1], for solving differential [2, 3] and integral equations [4], linear and nonlinear, has been the subject of extensive analytical and numerical studies.

The HPM is applied to singular nonlinear differential equations [5], nonlinear wave equations [6], nonlinear oscillators [7], bifurcation of delay-differential equations [8], boundary value problems [9], initial value problems [10], and nonlinear coupled equations [5, 11]. Furthermore the HPM yields every rapid convergence of the solution series in most cases.

On the other hand, the integral transformations played an essential role in many fields of science [12, 13], especially, engineering mathematics [14], mathematical physics [15], optics [16], image processing [17] and, few others because they have been successfully used in solving many problems in those fields. Many of these transforms have been used and applied on theory and applications, such as Sumudu [18, 19], Laplace [20, 21], Fourier [13], Elzaki et al. [22] and new integral transform [23]. Among these the most widely used is Laplace transform. Here, new integral transform is proposed to avoid the complexity of previous transforms [13, 18, 19].

In general, the nonlinear partial differential equations (NPDEs) have modeled nonlinear complex phenomena in various scientific fields [24–35]. The investigation of analytical, approximate, and exact solutions of NPDEs will help better understand the complex phynomena.

Our method, which is a coupling of the new integral transform and homotopy perturbation technniques, deforms continuously to a simple problem which is easily solved. Also this presentation has proposed a new method for solving NPDEs.

This article is organized as follows: In Section 2, we introduce some basic definitions and proberities for the new integral transform. In Section 3, we discuss the method used in this work. Some applications are given in Section 4 to show the accuracy and advantage of the proposed method. Finally, numerical results are discussed in Section 5.

2. Basic Definition of the New Integral Transform (NT)

In this section, we mention the following basic definitions and theorems of the new transform used in the present paper.

2.1. Definition of the New Transform. The transform of a function $f(t)$ is defined by

$$F(s) = T\{f(t)\} = \int_0^\infty e^{-t} e^{-t} f\left(\frac{t}{s}\right) dt, \quad s \in R. \tag{1}$$

Theorem 1. (Sufficient condition). *If a function h is piecewise continuous on* $[0, \infty)$ *and of exponential order so, then the transform of h exists for* $s > s_0$.

Theorem 2. (Linear combination). *If transforms* $T(u)$ *and* $T(v)$ *of in Equation (1) the functions* u, v *are well defined and* c_1, c_2 *are constants, then*

$$T\{c_1 u + c_2 v\} = c_1 T\{u\} + c_2 T\{v\}. \tag{2}$$

Theorem 3. (n^{th} Derivatives). *If the functions* $Tu, Tu', \ldots, Tu^{(n)}$ *are well defined,* $n = 1, 2, 3, \ldots,$ *then*

$$T\{u^n\} = s^n T\{u\} - \sum_{k=0}^{n-1} s^{n-k} u^{(k)}(0). \tag{3}$$

3. Analysis of the Method

To illustrate the modification algorithm of the NTHPM, we consider the following nonlinear partial differential equation with time derivatives of any order

$$L(u(x,t)) + N(u(x,t), u_x(x,t), u_{xx}(x,t)) = g(x,t). \tag{4}$$

where, L is linear differential operator $(L = d^n/dt^n)$, N represents the general nonlinear differential operator and $g(x,t)$ is the source term, subject to the initial conditions

$$\frac{\partial^m u(x,0)}{\partial t^m} = h_m(x), m = 0, 1, 2, \ldots, n-1. \tag{5}$$

In view of the homotopy technique, we can constract the following homotopy

$$\begin{aligned} H(u(x,t), p) &= (1-p)[L(u(x,t)) - L(u(x,0))] \\ &+ p[L(u(x,t)) + N(u(x,t)) - g(x,t)] = 0, \end{aligned} \tag{6}$$

where $p \in [0, 1]$· the homotopy parameter t always changes from zero to unity. The changing process of p is called deformation. When $p = 0$, Equation (6) becomes

$$L(u(x,t)) = L(u(x,0)), \tag{7}$$

and when $p = 1$, Equation (6) turns out to the original Equation (4). Since $u(x,0)$ is a function of x only, Equation (6) can be rewritten to be in the following form

$$\frac{\partial^n u(x,t)}{\partial t^n} + p[N(u(x,t)) - g(x,t)] = 0. \tag{8}$$

According to the homotopy technique, the basic assumption is that the solution of Equation (8) can be written as a power series in p as

$$u(x,t) = \sum_{i=0}^{\infty} p^i u_i(x,t), \tag{9}$$

where $u_i(x,t)$ are unknown functions to be determined. Now, taking in mind the initial conditions (2), the NT for Equation (8) gives

$$s^n T\{u(x,t)\} - \sum_{k=0}^{n-1} s^{n-k} u^k(x,0) + pT[N - g] = 0, \tag{10}$$

again taking the inverse of the NT for Equation (10), we obtain

$$u(x,t) - T^{-1}\left\{\sum_{k=0}^{n-1} \frac{1}{s^k} u^k(x,0)\right\} + T^{-1}\left\{\frac{1}{s^n} pT\{[N - g]\}\right\} = 0. \tag{11}$$

Substituting from Equation (9) into Equation (11), yields

$$\begin{aligned} &\sum_{i=0}^{\infty} p^i u_i(x,t) - T^{-1}\left\{\sum_{k=0}^{n-1} \frac{1}{s^k}\left(\sum_{i=0}^{\infty} p^i u_i^{(k)}(x,0)\right)\right\} \\ &+ T^{-1}\left\{\frac{1}{s^n} pT\left\{N\left(\sum_{i=0}^{\infty} p^i u_i(x,t)\right) - g(x,t)\right\}\right\} = 0. \end{aligned} \tag{12}$$

Equating the identical powers of p, therefore, after doing some calculations for the NT and the inverse of NT we get the unknown functions $u_0, u_1, u_2, \ldots$ After substituting into Equation (10) with $p = 1$, we get the solution of the problem (1)–(2).

4. Applications on NTHPM

Our method will be illustrated through examples in one-dimension for linear and nonlinear coupled systems of partial differential equations.

Example 1. Consider the one-dimensional linear system

$$u_{tt} - u_{xx} - \frac{1}{4}u - v + \frac{4}{5} = 0, \tag{13}$$

$$v_{tt} - v_{xx} - u - \frac{1}{4}v + \frac{4}{5} = 0, \tag{14}$$

subjected to the initial conditions

$$u(x,0) = 1 + e^x, u_t(x,0) = \frac{e^x}{2}, \tag{15}$$

$$v(x,0) = 1 - e^x, v_t(x,0) = -\frac{e^x}{2}. \tag{16}$$

Assume that the solutions of Equations (13) and (14) can be written as a power series as follows

$$u(x,t) = \sum_{i=0}^{\infty} p^i u_i(x,t), \tag{17}$$

$$v(x,t) = \sum_{i=0}^{\infty} p^i v_i(x,t), \tag{18}$$

substituting from Equation (17) into Equation (12) for $n = 2$, $R = -u_{xx} - (1/4)u - v$, $N = 0$ and $g(x,t) = 5/4$ and using the initial conditions (11), yields

$$\begin{aligned} &\sum_{i=0}^{\infty} p^i u_i(x,t) - T^{-1}\left\{1 + e^x + \frac{e^x}{2}\right\} \\ &+ T^{-1}\left\{\frac{1}{s^2} pT\left\{-\sum_{i=0}^{\infty} u_{ixx} - \frac{1}{4}\sum_{i=0}^{\infty} u_i - \sum_{i=0}^{\infty} v_i + \frac{5}{4}\right\}\right\} = 0, \end{aligned} \tag{19}$$

by a similar way, substituting Equation (18) into Equation (12), for $n = 2$, $R = -v_{xx} - (1/4)v - u, N = 0$ and $g(x,t) = 5/4$ and using the initial conditions (12), we obtain

$$\sum_{i=0}^{\infty} p^i v_i(x,t) - T^{-1}\left\{1 - e^x + \frac{e^x}{2}\right\} + T^{-1}\left\{\frac{1}{s^2}pT\left\{-\sum_{i=0}^{\infty} v_{ixx} - \frac{1}{4}\sum_{i=0}^{\infty} v_i - \sum_{i=0}^{\infty} v_i + \frac{5}{4}\right\}\right\} = 0, \quad (20)$$

On putting the coeffiecients to the power of p equal to zero in Equations (19), (20), we can obtain a series of linear equations, which are easy to solve by using Mathematica software to give

$$u_0 = T^{-1}\left\{1 + e^x + \frac{e^x}{2s}\right\} = 1 + e^x + \frac{e^x t}{2}, \quad (21)$$

$$v_0 = T^{-1}\left\{1 - e^x - \frac{e^x}{2s}\right\} = 1 - e^x - \frac{e^x t}{2}, \quad (22)$$

$$\begin{aligned} u_1 &= T^{-1}\left\{\frac{1}{s^2}T\left\{-\frac{5}{4} + \frac{u_0}{4} + v_0 + u_{0xx}\right\}\right\} \\ &= T^{-1}\left\{\frac{1}{s^2}T\left\{\frac{e^x}{4} + \frac{e^x t}{8}\right\}\right\} \\ &= T^{-1}\left\{\frac{e^x(1+2s)}{8s^3}\right\} = e^x\frac{\left(\frac{t}{2}\right)^2}{2!} + e^x\frac{\left(\frac{t}{2}\right)^3}{3!}, \end{aligned} \quad (23)$$

$$\begin{aligned} v_1 &= T^{-1}\left\{\frac{1}{s^2}T\left\{-\frac{5}{4} + u_0 + \frac{v_0}{4} + v_{0xx}\right\}\right\} \\ &= T^{-1}\left\{\frac{1}{s^2}T\left\{-\frac{1}{8}e^x(2+t)\right\}\right\} = T^{-1}\left\{-\frac{e^x(1+2s)}{8s^3}\right\} \\ &= -e^x\frac{\left(\frac{t}{2}\right)^2}{2!} - e^x\frac{\left(\frac{t}{2}\right)^3}{3!}, \end{aligned} \quad (24)$$

$$\begin{aligned} u_2 &= T^{-1}\left\{\frac{1}{s^2}T\left\{\frac{u_1}{4} + v_1 + u_{1xx}\right\}\right\} \\ &= T^{-1}\left\{\frac{1}{s^2}T\left\{\frac{1}{4}\left(\frac{e^x t^2}{8} + \frac{e^x t^3}{48}\right)\right\}\right\} \\ &= T^{-1}\left\{\frac{e^x(1+2s)}{32s^5}\right\} = e^x\frac{\left(\frac{t}{2}\right)^4}{4!} + e^x\frac{\left(\frac{t}{2}\right)^5}{5!}, \end{aligned} \quad (25)$$

$$\begin{aligned} v_2 &= T^{-1}\left\{\frac{1}{s^2}T\left\{\frac{u_1}{4} + \frac{v_1}{4} + v_{1xx}\right\}\right\} \\ &= T^{-1}\left\{\frac{1}{s^2}T\left\{-\frac{1}{4}\left(\frac{e^x t^2}{8} + \frac{e^x t^3}{48}\right)\right\}\right\} \\ &= T^{-1}\left\{-\frac{e^x(1+2s)}{32s^5}\right\} = -e^x\frac{\left(\frac{t}{2}\right)^4}{4!} - e^x\frac{\left(\frac{t}{2}\right)^5}{5!}, \end{aligned} \quad (26)$$

$$\begin{aligned} u_3 &= T^{-1}\left\{\frac{1}{s^2}T\left\{\frac{u_2}{4} + v_2 + u_{2xx}\right\}\right\} \\ &= T^{-1}\left\{\frac{1}{s^2}T\left\{e^x\left(\frac{t^4}{384} + \frac{t^5}{3840}\right)\right\}\right\} \\ &= T^{-1}\left\{\frac{e^x(1+2s)}{32s^5}\right\} = e^x\frac{\left(\frac{t}{2}\right)^4}{4!} + e^x\frac{\left(\frac{t}{2}\right)^5}{5!}, \end{aligned} \quad (27)$$

$$\begin{aligned} v_3 &= T^{-1}\left\{\frac{1}{s^2}T\left\{-u_2 + \frac{v_2}{4} + v_{2xx}\right\}\right\} \\ &= T^{-1}\left\{\frac{1}{s^2}T\left\{-e^x\left(\frac{t^4}{384} + \frac{t^5}{3840}\right)\right\}\right\} \\ &= T^{-1}\left\{-\frac{e^x(1+2s)}{32s^5}\right\} = -e^x\frac{\left(\frac{t}{2}\right)^4}{4!} - e^x\frac{\left(\frac{t}{2}\right)^5}{5!}, \end{aligned} \quad (28)$$

$$\begin{aligned} u_4 &= T^{-1}\left\{\frac{1}{s^2}T\left\{\frac{u_3}{4} + v_3 + u_{3xx}\right\}\right\} \\ &= T^{-1}\left\{\frac{1}{s^2}T\left\{\frac{1}{4}\left(\frac{e^x t^6}{46080} + \frac{e^x t^7}{645120}\right)\right\}\right\} \\ &= T^{-1}\left\{\frac{e^x(1+2s)}{512s^9}\right\} = e^x\frac{\left(\frac{t}{2}\right)^8}{8!} + e^x\frac{\left(\frac{t}{2}\right)^9}{9!}, \end{aligned} \quad (29)$$

$$\begin{aligned} v_4 &= T^{-1}\left\{\frac{1}{s^2}T\left\{u_3 + \frac{v_3}{4} + v_{3xx}\right\}\right\} \\ &= T^{-1}\left\{\frac{1}{s^2}T\left\{-\frac{1}{4}\left(\frac{e^x t^6}{46080} + \frac{e^x t^7}{645120}\right)\right\}\right\} \\ &= T^{-1}\left\{-\frac{e^x(1+2s)}{512s^9}\right\} = -e^x\frac{\left(\frac{t}{2}\right)^8}{8!} - e^x\frac{\left(\frac{t}{2}\right)^9}{9!}, \end{aligned} \quad (30)$$

and so on. Proceeding as before the rest of components were obtained, and then the two functions $u(x,t)$ and $v(x,t)$ in the closed form are readily found to be

$$u(x,t) = 1 + e^{x+\frac{t}{2}}, \quad (31)$$

$$v(x,t) = 1 - e^{x+\frac{t}{2}}. \quad (32)$$

Example 2. We consider the homogenuous form of coupled Burgers equations

$$u_t - u_{xx} - 2uu_x + (uv)_x = 0, \quad (33)$$

$$v_t - v_{xx} - 2vv_x + (uv)_x = 0, \quad (34)$$

with the initial conditions

$$u(x,0) = cosx, \quad (35)$$

$$v(x,0) = cosx. \quad (36)$$

As illustrated in Example 1, substituting from Equations (17) and (18) into Equation (12) but in this case for $n = 1$ and using the initial conditions (21) and (22) respectively, we get

$$\sum_{i=0}^{\infty} p^i u_i(x,t) - T^{-1}\{cosx\} - T^{-1}\left\{\frac{1}{s}pT\left\{\sum_{i=0}^{\infty} u_{ixx} + 2\sum_{i=0}^{\infty} u_i \sum_{i=0}^{\infty} u_{ix} + \left(\sum_{i=0}^{\infty} u_i \sum_{i=0}^{\infty} v_i\right)_x\right\}\right\} = 0, \quad (37)$$

$$\sum_{i=0}^{\infty} p^i v_i(x,t) - T^{-1}\{cosx\} - T^{-1}\left\{\frac{1}{s}pT\left\{\sum_{i=0}^{\infty} v_{ixx} + 2\sum_{i=0}^{\infty} v_i \sum_{i=0}^{\infty} v_{ix} + \left(\sum_{i=0}^{\infty} u_i \sum_{i=0}^{\infty} v_i\right)_x\right\}\right\} = 0. \quad (38)$$

Putting the coefficients of the power of p equal to zero in Equations (37) and (38), we obtain

$$u_0 = T^{-1}\{cosx\} = cosx, \quad v_0 = T^{-1}\{cosx\} = cosx, \quad (39)$$

$$\begin{aligned} u_1(x,t) &= T^{-1}\left\{\frac{1}{s}T\{2u_0u_{0x} - v_0u_{0x} - u_0v_{0x} + u_{0xx}\}\right\} \\ &= T^{-1}\left\{-\frac{1}{s}cosx\right\} = -tcosx, \end{aligned} \quad (40)$$

$$\begin{aligned} v_1(x,t) &= T^{-1}\left\{\frac{1}{s}T\{2v_0v_{0x} - u_0v_{0x} - v_0u_{0x} + v_{0xx}\}\right\} \\ &= T^{-1}\left\{-\frac{1}{s}cosx\right\} = -tcosxx, \end{aligned} \quad (41)$$

$$\begin{aligned} u_2(x,t) &= T^{-1}\left\{\frac{1}{s}T\{2u_1u_{0x} - v_1u_{0x} + 2u_0u_{1x} - v_0u_{1x} \right. \\ &\quad \left. -u_1v_{0x} - u_0v_{1x} + u_{1xx}\}\right\} \\ &= T^{-1}\left\{\frac{1}{s}T\{tcosx\}\right\} = T^{-1}\left\{\frac{1}{s^2}cosx\right\} = \frac{t^2}{2!}cosx, \end{aligned} \quad (42)$$

$$\begin{aligned} v_2(x,t) &= T^{-1}\left\{\tfrac{1}{s}T\{2v_1v_{0x} - u_1v_{0x} + 2v_0v_{1x} - u_0v_{1x} \right. \\ &\quad \left. -v_1u_{0x} - v_0u_{1x} + v_{1xx}\}\right\} \\ &= T^{-1}\left\{\tfrac{1}{s}T\{tcosx\}\right\} = T^{-1}\left\{\tfrac{1}{s^2}cosx\right\} = \tfrac{t^2}{2!}cosx, \end{aligned} \quad (43)$$

$$\begin{aligned} u_3(x,t) &= T^{-1}\left\{\frac{1}{s}T\{2u_2u_{0x} - v_2u_{0x} + 2u_1u_{1x} - v_1u_{1x} \right. \\ &\quad \left. + 2u_0u_{2x} - v_0u_{2x} - u_2v_{0x} - u_1v_{1x} - u_0v_{2x} - u_{2xx}\}\right\} \\ &= T^{-1}\left\{\frac{1}{s}T\left\{\frac{t^2}{2}cosx\right\}\right\} = T^{-1}\left\{\frac{1}{s^3}cosx\right\} = \frac{t^3}{3!}cosx. \end{aligned} \quad (44)$$

$$\begin{aligned} v_3(x,t) &= T^{-1}\left\{\frac{1}{s}T\{v_2u_{0x} + v_1u_{1x} + v_0u_{2x} + u_2v_{0x} - 2v_2v_{0x} \right. \\ &\quad \left. + u_1v_{1x} - 2v_1v_{1x} + u_0v_{2x} - 2v_0v_{2x} + v_{2xx}\}\right\} \\ &= T^{-1}\left\{\frac{1}{s}T\left\{\frac{-t^2}{2}cosx\right\}\right\} = T^{-1}\left\{\frac{-1}{s^3}cosx\right\} = \frac{-t^3}{3!}cosx. \end{aligned} \quad (45)$$

$$\begin{aligned} u_4(x,t) &= T^{-1}\left\{\frac{1}{s}T\{2u_3u_{0x} - v_3u_{0x} + 2u_2u_{1x} - v_2u_{1x} + 2u_1u_{2x} - v_1u_{2x} \right. \\ &\quad \left. + 2u_0u_{3x} - v_0u_{3x} - u_3v_{0x} - u_2v_{1x} - u_1v_{2x} - u_0v_{3x} + u_{3xx}\}\right\} \\ &= T^{-1}\left\{\frac{1}{s}T\left\{\frac{t^3}{6}cosx\right\}\right\} = T^{-1}\left\{\frac{1}{s^4}cosx\right\} = \frac{t^4}{4!}cosx. \end{aligned} \quad (46)$$

$$\begin{aligned} v_4(x,t) &= T^{-1}\left\{\frac{1}{s}T\{2v_3v_{0x} - u_3v_{0x} + 2v_2v_{1x} - u_2v_{1x} + 2v_1v_{2x} - u_1v_{2x} \right. \\ &\quad \left. + 2v_0v_{3x} - u_0v_{3x} - v_3u_{0x} - v_2u_{1x} - v_1u_{2x} - v_0u_{3x} + v_{3xx}\}\right\} \\ &= T^{-1}\left\{\frac{1}{s}T\left\{\frac{t^3}{6}cosx\right\}\right\} = T^{-1}\left\{\frac{1}{s^4}cosx\right\} = \frac{t^4}{4!}cosx. \end{aligned} \quad (47)$$

again substituting the functions $u_i, v_i, i = 0, 1, 2, \ldots$ into the Equations (17) and (18), we obtain directly

$$u(x,t) = v(x,t) = cosx\left(1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!}\right). \quad (48)$$

Furthermore, in closed form Equation (48) takes the form

$$u(x,t) = v(x,t) = e^{-t}\cos x. \quad (49)$$

Example 3. Next, we consider the nonlinear Drinfeld Sokolov system in the following form

$$u_t + \left(v^2\right)_x = 1 - 2(t - x), \quad (50)$$

$$v_t - v_{xxx} + (uv)_x = 1 - 2x, \quad (51)$$

with the initial conditions

$$u(x,0) = x, v(x,0) = -x. \quad (52)$$

By applying the same steps used in Examples (1) and (2), we can easily get

$$u_0 = T^{-1}\{x\} = x, v_0 = T^{-1}\{-x\} = -x, \quad (53)$$

$$\begin{aligned} u_1(x,t) &= T^{-1}\left\{\frac{1}{s}T\{1 - 2t + 2x - 2v_0v_{0x}\}\right\} \\ &= T^{-1}\left\{\frac{1}{s}T\{1 - 2t\}\right\} = T^{-1}\left\{\frac{1}{s}\left(1 - \frac{2}{s}\right)\right\} = t - t^2, \end{aligned} \quad (54)$$

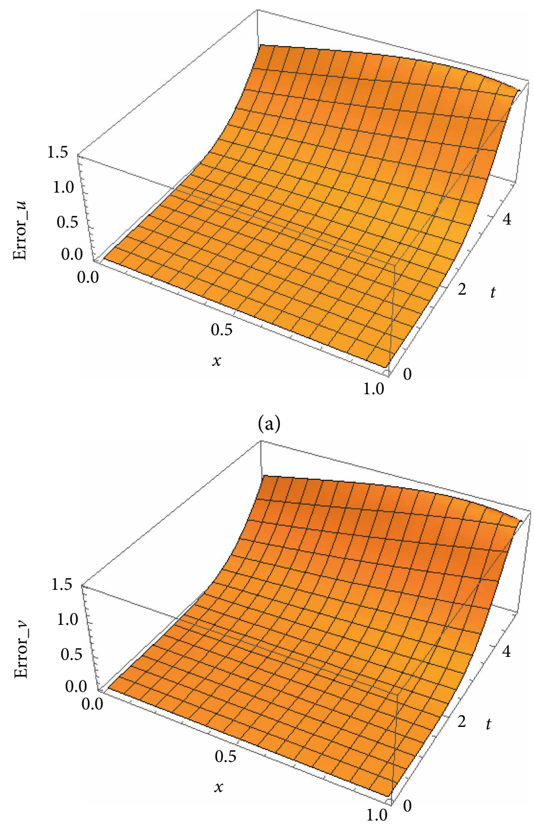
$$\begin{aligned} v_1(x,t) &= T^{-1}\left\{\frac{1}{s}T\{1 - 2x - v_0u_{0x} - u_0v_{0x} + v_{0xxx}\}\right\} \\ &= T^{-1}\left\{\frac{1}{s}T\{1\}\right\} = T^{-1}\left\{\frac{1}{s}\right\} = t, \end{aligned} \quad (55)$$

$$\begin{aligned} u_2(x,t) &= T^{-1}\left\{\frac{1}{s}T\{-2v_1v_{0x} - 2v_0v_{1x}\}\right\} = T^{-1}\left\{\frac{1}{s}T\{2t\}\right\} \\ &= T^{-1}\left\{\frac{1}{s}\left(\frac{2}{s}\right)\right\} = 2T^{-1}\left\{\frac{1}{s^2}\right\} = t^2. \end{aligned} \quad (56)$$

$$\begin{aligned} v_2(x,t) &= T^{-1}\left\{\frac{1}{s}T\{-v_1u_{0x} - v_0u_{1x} - u_1v_{0x} - u_0v_{1x} + v_{1xxx}\}\right\} \\ &= -T^{-1}\left\{\frac{1}{s}T\left[t^2\right]\right\} = -T^{-1}\left\{\frac{1}{s}\left(\frac{2}{s}\right)\right\} = -2T^{-1}\left\{\frac{1}{s^2}\right\} = -\frac{t^3}{3}. \end{aligned} \quad (57)$$

$$\begin{aligned} u_3(x,t) &= T^{-1}\left\{\frac{1}{s}T\{2v_2v_{0x} + 2v_1v_{1x} + 2v_0v_{2x}\}\right\} \\ &= T^{-1}\left\{\frac{1}{s}T\left\{\frac{-2t^3}{3}\right\}\right\} = -T^{-1}\left\{\frac{4}{s^4}\right\} = -\frac{t^4}{6}. \end{aligned} \quad (58)$$

$$\begin{aligned} v_3(x,t) &= T^{-1}\left\{\frac{1}{s}T\{v_2u_{0x} + v_1u_{1x} + v_0u_{2x} + u_2v_{0x} \right. \\ &\quad \left. + u_1v_{1x} + u_0v_{2x} - u_{2xxx}\}\right\} \\ &= T^{-1}\left\{\frac{1}{s}T\left\{t^2 + \frac{t^3}{3}\right\}\right\} = T^{-1}\left\{\frac{1}{s}\left(\frac{2!}{s^3} + \frac{3!}{3s^3}\right)\right\} \\ &= \frac{t^3}{3} + \frac{t^4}{12}. \end{aligned} \quad (59)$$



(a)

(b)

Figure 1: The figures explains the surface errors for Example (4.1). (a) Error-$u = |Ex \cdot u - App \cdot u|$. (b) Error-$v = |Ex \cdot v - App \cdot v|$.

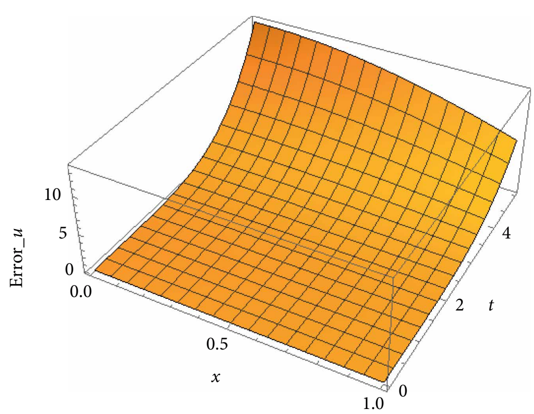
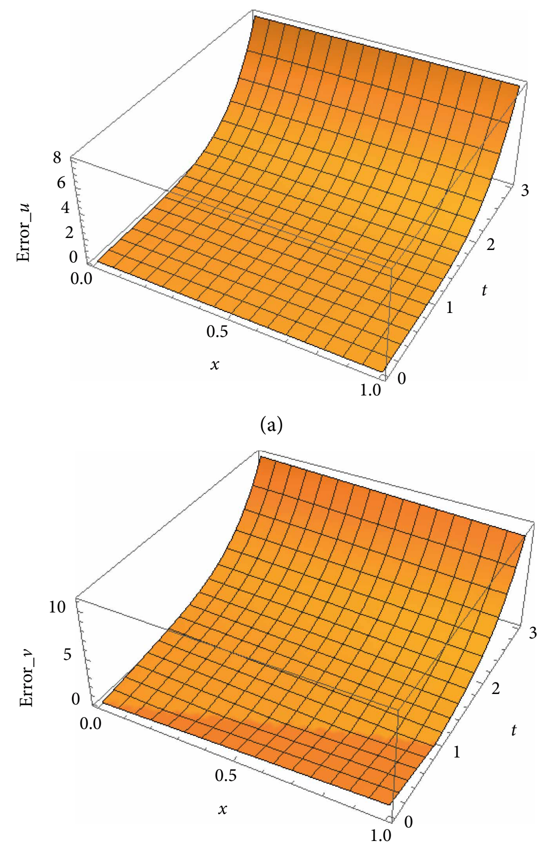


Figure 2: The figure explain the surface errors $u = v$ for Example (4.2).

$$\begin{aligned} u_4(x,t) &= T^{-1}\left\{\tfrac{1}{s}T\{2v_3v_{0x} + 2v_2v_{1x} + 2v_1v_{2x} + 2v_0v_{3x}\}\right\} \\ &= T^{-1}\left\{\tfrac{1}{s}T\left\{2\left(\tfrac{t^3}{3} + \tfrac{t^4}{12}\right)\right\}\right\} \\ &= 4T^{-1}\left\{\tfrac{1}{s^4} + \tfrac{1}{s^5}\right\} = \tfrac{t^4}{6} + \tfrac{t^5}{30}. \end{aligned} \tag{60}$$

$$\begin{aligned} v_4(x,t) = T^{-1}\bigg\{\frac{1}{s}T\{&v_3u_{0x} + v_2u_{1x} + v_1u_{2x} + v_0u_{3x} \\ &+u_3v_{0x} + u_2v_{1x} + u_1v_{2x} + u_0v_{3x} - v_{3xxx}\}\bigg\} \\ = -T^{-1}&\left\{\frac{1}{s}T\left\{\frac{t^3}{3s} + \frac{t^4}{4s}\right\}\right\} \\ = -T^{-1}&\left\{\left(\frac{3!}{3s^4} + \frac{4!}{4s^5}\right)\right\} = -\frac{t^4}{12} - \frac{t^5}{30}. \end{aligned} \tag{61}$$



(a)

(b)

Figure 3: The figures explains the surface errors for Example (4.3). (a) Error-$u = |Ex \cdot u - App \cdot u|$. (b) Error-$v = |Ex \cdot v - App \cdot v|$.

Substituting into Equation (17) and (18), we obtain

$$u(x,t) = x + t, \tag{62}$$

$$v(x,t) = x - t. \tag{63}$$

5. Numerical Results and Discussion

The numerical behavior of the error between the exact solution and the solution obtained by NTHPM is shown in Figures 1–3. The numerical results are obtained by using fourth order perturbation only from the series formulas (17), (18) with $p = 1$. From these figures, we achieved a very good approximation for the solution of our systems at the small values of time t, but at the large values of the time, the error can be reduced by adding new terms from the iteration formulas.

6. Conclusion

In this work, we have proposed a modification to the homotopy perturbation technique by combining it with a new integral transform. The aim of this approach is to obtain exact solutions of linear as well as nonlinear coupled systems. The efficiency and accuracy of the present scheme are validated through some examples. The results show that NITHPM is a powerful and good technique for obtaining exact solutions of many systems of linear as well as nonlinear differential equations. The computations associated in this work are performed by using the Mathematica software.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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