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Research Article

Some Curvature Properties on Lorentzian Generalized Sasakian-Space-Forms

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In this paper, we investigate the Lorentzian generalized Sasakian-space-form. We give the necessary and sufficient conditions for the Lorentzian generalized Sasakian-space-form to be projectively flat, conformally flat, conharmonically flat, and Ricci semisymmetric and their relationship between each other. As the application of our theorems, we study the Ricci almost soliton on conformally flat Lorentzian generalized Sasakian-space-form.

1. Introduction

Gauge theory, as we all know, has a lot of profound intension and it has permeated all aspects of theoretical physics. It will surely guide future developments in theoretical physics. Gauge theory and principal fiber bundle theory are inextricably linked with each other (see [1]). For instance, the field strength $f^{\kappa}_{\mu\nu}$ of gauge theory is exactly the curvature of a manifold (see [2]). So if we know the curvature properties of a manifold, we can get the distribution of field strength $f^{\kappa}_{\mu\nu}$. The purpose of our paper is to clarify the unsteady field around Lorentzian generalized Sasakian-space-forms in view of principal fiber bundle theory.

In differential geometry, the curvature tensor R is very significant to the nature of a manifold. Many other curvature tensor fields defining on the manifold are related with curvature tensor, for instance, Ricci tensor S, scalar curvature r, and conharmonic curvature tensor K. It has been proven that the curvature depends on sectional curvatures entirely. If a manifold is of constant sectional curvature, then we call it a space-form.

For a *Sasakian manifold*, we have the definition of ϕ-sectional curvature and it plays the same role as a sectional curvature. If the ϕ-sectional curvature of a Sasakian manifold is constant, then the manifold is a *Sasakian-space-form* (see [3]). As a generalization of Sasakian-space-form,-*generalized Sasakian-space-form* was introduced and investigated in [4] and the authors also gave some examples. In short, a generalized Sasakian-space-form is an almost contact metric manifold that the curvature tensor R is related with three smooth functions f_1, f_2, and f_3 defined on the manifold.

In [5], the authors defined the *generalized indefinite Sasakian-space-form*. It is the generalized Sasakian-space-form with a semi-Riemannian metric. In this paper, we are most interested in the Lorentzian manifold because it is very useful in Einstein's general relativity. We call it *Lorentzian generalized Sasakian-space-form*, and to make our paper more concise, we will write it as LGSSF for short. We give the necessary and sufficient condition of the LGSSF with the dimension equal to or greater than five to be some certain curvature tensor conditions. We also clarify the necessary and sufficient condition that LGSSF is *Ricci semisymmetric*. It is meaningful to dig into LGSSF satisfying these conditions because we can understand the relationship between the functions f_1, f_2, and f_3 and the curvature properties of the manifold.

Ricci flow is a powerful tool to investigate manifolds. It was first introduced by Hamilton in [6], and he used it to investigate Riemannian manifolds with positive curvature. There are many solutions to Ricci flow, and the *Ricci soliton* is the self-similar solution of it. Physicists are also interested in the Ricci soliton because in physics, it is regarded as a quasi-Einstein metric. In our paper, we give the Ricci soliton equation as follows:

$$L_W g + 2S = 2\lambda g. \quad (1)$$

In the equation, L_W denotes the Lie derivative, S denotes the Ricci tensor, g denotes the Riemannian metric, and λ is a real scalar. We call it the triple (g, W, and λ) Ricci soliton on the manifold. People can also use the Ricci soliton to study semi-Riemannian manifolds and refer to [7–9] for more details.

In [10], Pigola et al. introduced and studied the *Ricci almost soliton*. They replaced the real scalar λ by a smooth function defining the manifold and called it the triple (g, W, and λ) Ricci almost soliton. In our paper, we apply the Ricci almost soliton to LGSSF, and in consideration of the curvature properties of the manifolds, we get some interesting results.

We organize our paper as follows. In Section 2, readers can get several basic definitions about LGSSF. Sections 3, 4, 5, and 6 are dedicated to showing how a LGSSF can be projectively flat, conformally flat, conharmonically flat, and Ricci semisymmetric. In Section 7, we apply what we get from Sections 3, 4, 5, and 6 to a Ricci almost soliton on LGSSF and give two examples.

We use U, W, V, X, Y, and Z to denote the smooth tangent vector fields on the manifold, and all manifolds and functions mentioned in our paper are smooth.

2. Preliminaries

If a semi-Riemannian manifold M admits a vector field ζ (we call it a *Reeb vector field* or *characteristic vector field*), a 1-form η, and a (1,1) tensor field ϕ satisfying

$$\begin{aligned} \phi\zeta &= 0, \\ \eta\circ\phi &= 0, \\ \phi^2 &= -id + \eta\otimes\zeta, \\ \eta(\zeta) &= 1, \\ \eta(U) &= \varepsilon g(\zeta, U), \\ g(U, W) &= g(\phi U, \phi W) + \varepsilon\eta(U)\eta(W), \end{aligned} \quad (2)$$

where $\varepsilon = g(\zeta,\zeta) = \pm 1$, then we call such a manifold an ε-almost contact metric manifold [11] or almost contact pseudometric manifold [12], and we call it the triple (ϕ, ζ, and η) almost contact structure on the manifold.

If the 2-form $d\eta$ and the metric g satisfy

$$d\eta(U, W) = g(U, \phi W), \quad (3)$$

then the manifold M is a contact pseudometric manifold and the triple (ϕ, ζ, and η) is a contact structure on the manifold.

We define a vector field on the product $\mathbb{R}\times M^{2n+1}$ by $(h(d/dx), U)$; x is the coordinate on $\mathbb{R}$ and h is a C^∞ function on $\mathbb{R}\times M^{2n+1}$. We define an almost complex structure J on $\mathbb{R}\times M^{2n+1}$ by

$$J\left(h\frac{d}{dx}, U\right) = \left(\eta(U)\frac{d}{dx}, \phi U - h\zeta\right), \quad (4)$$

and it is easy to check $J^2 = -id$. Moreover, if J is integrable, then we will say the almost contact structure (ϕ, ζ, and η) is normal (see [3]). We call an ε-normal contact metric manifold an indefinite Sasakian manifold or an ε-Sasakian manifold.

Now we give the definition of the ϕ-sectional curvature. The plane spanned by U and ϕU is called ϕ-section if U is orthogonal to ζ. The ϕ-sectional curvature is the sectional curvature $K(U, \phi U)$. The curvature of an indefinite Sasakian manifold is determined by ϕ-sectional curvatures entirely.

If the ϕ-sectional curvature of an ε-Sasakian manifold is a constant c, then the curvature tensor of the manifold has the following form [13]:

$$\begin{aligned} R(U, W)X = {} & \frac{c+3\varepsilon}{4}\{g(W, X)U - g(U, X)W\} \\ & + \frac{c-\varepsilon}{4}\{g(U, \phi X)\phi W - g(W, \phi X)\phi U \\ & + 2g(U, \phi W)\phi X\} + \frac{c-\varepsilon}{4}\{\eta(U)\eta(X)W \\ & - \eta(W)\eta(X)U + \varepsilon g(U, X)\eta(W)\zeta \\ & - \varepsilon g(W, X)\eta(U)\zeta\}. \end{aligned} \quad (5)$$

In [5], the author replaced the constants with three smooth functions defining the manifold. For an ε-almost contact metric manifold M, if the curvature tensor is given by

$$\begin{aligned} R(U, W)X = {} & f_1\{g(W, X)U - g(U, X)W\} \\ & + f_2\{g(U, \phi X)\phi W - g(W, \phi X)\phi U \\ & + 2g(U, \phi W)\phi X\} + f_3\{\eta(U)\eta(X)W \\ & - \eta(W)\eta(X)U + \varepsilon g(U, X)\eta(W)\zeta \\ & - \varepsilon g(W, X)\eta(U)\zeta\}, \end{aligned} \quad (6)$$

where $f_1, f_2, f_3 \in C^\infty(M)$, then we call M the generalized indefinite Sasakian-space-form.

In our paper, we only focus on the Lorentzian situation: $\varepsilon = -1$ and the index of the metric is one. We call such manifold the Lorentzian generalized Sasakian-space-form, and in our paper, we denote it by $M_1^{2n+1}(f_1, f_2, f_3)$. Because some of the curvature tensor fields we studied are not suitable for three manifolds, in the following, the dimension of LGSSF $M_1^{2n+1}(f_1, f_2, f_3)$ is greater than three, that is, $n > 1$.

For a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)$, we have two useful equations from (6):

$$R(U, W)\zeta = (f_1 + f_3)(\eta(U)W - \eta(W)U), \quad (7)$$

$$R(\zeta, U)W = (f_1 + f_3)(g(U, W)\zeta + \eta(W)U). \quad (8)$$

Lemma 1. *For a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)$, the Ricci tensor S is*

$$S(U,W) = (2nf_1 + 3f_2 + f_3)g(U,W) + (3f_2 - (2n-1)f_3)\eta(U)\eta(W), \quad (9)$$

so the Ricci operator Q and scalar curvature r are

$$QU = (2nf_1 + 3f_2 + f_3)U + ((2n-1)f_3 - 3f_2)\eta(U)\zeta, \quad (10)$$

$$r = 2n(2n+1)f_1 + 6nf_2 + 4nf_3. \quad (11)$$

Proof. As we all know for a semi-Riemannian manifold of dimension n, the Ricci tensor S and the scalar curvature r are

$$S(U,W) = \sum_{i=1}^{n} \varepsilon_i g(R(U,E_i)E_i, W), \quad r = \sum_{i=1}^{n} \varepsilon_i S(E_i, E_i), \quad (12)$$

where $\{E_i, \cdots, E_n\}$ is a local orthonormal frame field on the manifold and ε_i is the signature of E_i. The curvature tensor of $M_1^{2n+1}(f_1, f_2, f_3)$ is given by (6) and we know $g(U,W) = \sum \varepsilon_i g(U,E_i)g(X,E_i)$, so we can easily get (9), (10), and (11).

We can use warped product to construct LGSSF (see [5]). Let $h > 0$ be a function on $\mathbb{R}$ and $(N^{2n}, J, \text{and } G)$ be an almost complex manifold. Then, the warped product $M = \mathbb{R} \times_h N$ is a LGSSF with the Lorentzian metric given by

$$g_h = -\pi^*(g_{\mathbb{R}}) + (h \circ \pi)^2 \sigma^*(G), \quad (13)$$

where π is the projection from $\mathbb{R} \times N$ to $\mathbb{R}$ and σ is the projection to N. The almost contact structure is

$$\zeta = \frac{\partial}{\partial x}, \quad \eta(U) = -g_h(U,\zeta), \quad \phi(U) = (J\sigma_* U)^*. \quad (14)$$

Theorem 2 (see [5]). *Given a generalized complex space-form $N^{2n}(F_1, F_2)$. Then, $M_1^{2n+1}(f_1, f_2, f_3) = \mathbb{R} \times_h N$ is LGSSF, with functions*

$$f_1 = \frac{(F_1 \circ \pi) + h'^2}{h^2}, \quad f_2 = \frac{F_2 \circ \pi}{h^2}, \quad f_3 = -\frac{(F_1 \circ \pi) + h'^2}{h^2} + \frac{h''}{h}. \quad (15)$$

3. Projectively Flat Lorentzian Generalized Sasakian-Space-Form

For a $(2n+1)$-dimensional $(n > 1)$ smooth manifold M, the *projective curvature tensorP* is defined by

$$P(U,W)X = \frac{1}{2n}\{S(U,X)W - S(W,X)U\} + R(U,W)X. \quad (16)$$

It is a way to measure whether a manifold is a space-form because if M is projectively flat $(P = 0)$, then it must be of constant curvature and the converse is also true. For more details, readers can refer to [14].

Theorem 3. *A LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is projectively flat if and only if $f_2 = f_3 = 0$.*

Proof. Firstly, we suppose that $P(U,W)X = 0$. Put $U = \zeta$ and replace X by ϕX, then equation (16) will be

$$P(\zeta, W)\phi X = \frac{1}{2n}((2n-1)f_3 - 3f_2)g(W, \phi X)\zeta = 0. \quad (17)$$

In consideration of $g(W, \phi X) \neq 0$, we have

$$(2n-1)f_3 - 3f_2 = 0. \quad (18)$$

Then, equation (9) will be

$$S(W,U) = (2nf_1 + 3f_2 + f_3)g(W,U) = 2n(f_1 + f_3)g(W,U). \quad (19)$$

By the above equation, we can write (16) as

$$\begin{aligned} g(P(U,W)X,Z) = {} & f_2\{g(U,\phi X)g(\phi W,Z) - g(W,\phi X)g(\phi U,Z) + 2g(U,\phi W)g(\phi X,Z)\} \\ & - f_3\{\eta(W)\eta(X)g(U,Z) - \eta(U)\eta(X)g(W,Z) + \eta(W)\eta(Z)g(U,X) \\ & - \eta(U)\eta(Z)g(W,X) + g(W,X)g(U,Z) - g(U,X)g(W,Z)\} = 0. \end{aligned} \quad (20)$$

Setting $U = \phi U$ and $W = \phi W$, we have

$$\begin{aligned} g(P(\phi U, \phi W)X, Z) = {} & f_2\{g(\phi U, \phi X)g(\phi^2 W, Z) + 2g(\phi U, \phi^2 W)g(\phi X, Z) - g(\phi W, \phi X)g(\phi^2 U, Z)\} \\ & + f_3\{g(\phi U, X)g(\phi W, Z) - g(\phi W, X)g(\phi U, Z)\} = 0. \end{aligned} \quad (21)$$

Let us denote the orthonormal local basis of TM by $\{e_1, \cdots, e_{2n}, e_{2n+1} = \zeta\}$. Obviously, the signature of the local basis is $\{+, \cdots, +, -\}$ and we denote it by $\{\varepsilon_1, \cdots, \varepsilon_{2n}, \varepsilon_{2n+1}\}$. Putting $W = e_i$ and $Z = \varepsilon_i e_i$ in the above equation and summing over i, we will have the following equation:

$$(f_3 - (2n+1)f_2)g(\phi U, \phi X) = 0, \tag{22}$$

since $g(\phi U, \phi X) = \sum_{i=1}^{2n+1} \varepsilon_i g(\phi U, e_i) g(\phi X, e_i)$. Because of $g(\phi U, \phi X) \neq 0$, we get

$$f_3 - (2n+1)f_2 = 0. \tag{23}$$

Taking consideration of $(2n-1)f_3 - 3f_2 = 0$ and $n > 1$, we get

$$f_2 = f_3 = 0. \tag{24}$$

Conversely, we suppose that $f_2 = f_3 = 0$ then use (6) and (9), then (16) will be

$$\begin{aligned} P(U, W)X &= f_1\{g(U, X)W - g(W, X)U\} \\ &\quad - f_1\{g(U, X)W - g(W, X)U\} = 0. \end{aligned} \tag{25}$$

In order to get the next theorem of our paper, we first introduce the following famous theorem.

Schur.Theorem (see [15]). If $M^n (n \geq 3)$ is a connected semi-Riemannian manifold, and for each $m \in M$, the sectional curvature $K(m)$ is a constant function on the nondegenerate planes in $T_m M$, then $K(m)$ is a constant function on the manifold.

From Theorem 3, we can get if a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)$ is projectively flat, then $K(m) = f_1$. Using Schur.Theorem, we have the following theorem.

Theorem 4. *If a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is projectively flat, then f_1 is a constant function.*

4. Conformally Flat Lorentzian Generalized Sasakian-Space-Form

The *conformal curvature tensor C* is an important curvature tensor for a manifold, apart from the projective curvature tensor. For a $(2n+1)$-dimensional $(n > 1)$ smooth manifold, it is given by

$$\begin{aligned} C(U, W)X &= \frac{1}{2n-1}\{S(U, X)W - S(W, X)U \\ &\quad + g(U, X)QW - g(W, X)QU\} \\ &\quad + \frac{r}{2n(2n-1)}\{g(W, X)U - g(U, X)W\} \\ &\quad + R(U, W)X. \end{aligned} \tag{26}$$

Conformal curvature tensor C is the invariant of conformal transformation. In gauge field theory, it is used to classify the regular form of a curvature tensor when $S(e_i, e_j) \neq 0$. If the metric of a manifold is conformally related with a flat metric, then we will say the manifold is conformally flat $(C = 0)$.

Theorem 5. *A LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is conformally flat if and only if $f_2 = 0$.*

Proof. From (6), (9), (10), and (11), equation (26) becomes

$$\begin{aligned} C(U, W)X &= f_2\{g(X, \phi W)\phi U - g(X, \phi U)\phi W \\ &\quad + 2g(U, \phi W)\phi X\} \\ &\quad - \frac{3}{2n-1}f_2\{g(W, X)U - g(U, X)W \\ &\quad + \eta(W)\eta(X)U - \eta(U)\eta(X)W \\ &\quad + g(U, X)\eta(W)\zeta - g(W, X)\eta(U)\zeta\}. \end{aligned} \tag{27}$$

So if $f_2 = 0$, then C is zero.

Conversely, we suppose that $C(U, W)X = 0$; first, we put $U = \phi W$ in the above equation, then we will have

$$\begin{aligned} C(U, W)X &= 3f_2\{g(\phi W, X)W - g(W, X)\phi W \\ &\quad - \eta(W)\eta(X)\phi W - g(\phi W, X)\eta(W)\zeta\} \\ &\quad + (2n-1)f_2\{g(X, \phi W)(-W + \eta(W)\zeta) \\ &\quad + g(W, X)\phi W + \eta(U)\eta(W)\phi W \\ &\quad + 2g(W, W)\phi X + 2\eta(W)\eta(W)\phi X\} \\ &= 3f_2 g(\phi W, X)W - 3f_2\eta(W)\eta(X)\phi W \\ &\quad - 3f_2 g(W, X)\phi W \\ &\quad - 3f_2 g(\phi W, X)\eta(W)\zeta(2n-1)f_2 \\ &\quad \cdot \{g(X, \phi W)\eta(W)\zeta - g(X, \phi W)W \\ &\quad + g(W, X)\phi W + \eta(X)\eta(W)\phi W + 2g(W, W) \\ &\quad + 2\eta(W)\eta(W)\phi X\} = 0. \end{aligned} \tag{28}$$

Then, we have

$$\begin{aligned} &(n-2)f_2\{g(\phi W, X)W - g(W, X)\phi W - g(X, \phi W)\eta(W)\zeta \\ &\quad - \eta(W)\eta(X)\phi W\} - (2n-1)f_2\{\eta(W)\eta(W)\phi X \\ &\quad + g(W, W)\phi X\} = 0. \end{aligned} \tag{29}$$

Again we use the local orthonormal basis $\{e_1, \cdots, e_{2n}, e_{2n+1} = \zeta\}$ with signature $\{\varepsilon_1, \cdots, \varepsilon_{2n}, \varepsilon_{2n+1} = \varepsilon\}$; we choose $X = W = e_k (1 \leq k \leq 2n)$, so $g(W, \zeta) = g(X, \zeta) = 0$ and the above equation becomes

$$(n-2)f_2\varepsilon_k\phi e_k + (2n-1)f_2\varepsilon_k\phi e_k = 0, \tag{30}$$

thus, we have

$$(n-1)f_2\phi e_k = 0. \tag{31}$$

Because n is greater than one, we get $f_2 = 0$.

From Theorem 3, we can get the following theorem.

Theorem 6. *If a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is projectively flat, then it is conformally flat.*

5. Conharmonically Flat Lorentzian Generalized Sasakian-Space-Form

The conharmonic transformation is a kind of special conformal transformation. In general, a conformal transformation does not preserve the harmonic function defined on the manifold. In [16], Ishii introduced and studied the conharmonic transformation, which preserved a special kind of harmonic function. He also proved that a manifold could be reduced to a flat space by a conharmonic transformation if and only if the conharmonic curvature tensor K vanished everywhere on the manifold. In other words, the manifold is conharmonically flat ($K = 0$). For a $(2n + 1)$-dimensional ($n > 1$) smooth manifold, the *conharmonic curvature tensorK* is given by

$$K(U, W)X = \frac{1}{2n-1}\{g(U, X)QW - g(W, X)QU + S(U, X)W - S(W, X)U\} + R(U, W)X. \quad (32)$$

Definition 7. A $(2n + 1)$-dimensional ($n > 1$) LGSSF is said to be ζ-conharmonically flat if it satisfies

$$K(U, W)\zeta = 0. \quad (33)$$

Lemma 8. *A LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is ζ-conharmonically flat if and only if $(2n + 1)f_1 + 3f_2 + 2f_3 = 0$.*

Proof. From (7) and (10), equation (33) becomes

$$\begin{aligned} K(U, W)\zeta = {} & \frac{1}{2n-1}\{2n(f_1 + f_3)\eta(W)U \\ & - 2n(f_1 + f_3)\eta(U)W \\ & + (2nf_1 + 3f_2 + f_3)\eta(W)U \\ & - (2nf_1 + 3f_2 + f_3)\eta(U)W\} \\ & + (f_1 + f_3)\{\eta(U)W - \eta(W)U\} \\ = {} & \frac{1}{2n-1}((2n-1)f_1 + 3f_2 + 2f_3)\{\eta(W)U - \eta(U)W\}. \end{aligned} \quad (34)$$

So $M_1^{2n+1}(f_1, f_2, f_3)$ is ζ-conharmonically flat if and only if $(2n + 1)f_1 + 3f_2 + 2f_3 = 0$.

From equation (11) and Lemma 8, we have the following theorem.

Theorem 9. *A LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is ζ-conharmonically flat if and only if its scalar curvature $r = 0$.*

By Theorem 3 and Lemma 8, we have the following theorem.

Theorem 10. *If a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is ζ-conharmonically flat and projectively flat, then it is a flat manifold.*

We know that being conharmonically flat is the sufficient condition of ζ-conharmonically flat. So we have the following theorem.

Theorem 11. *If a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is conharmonically flat and projectively flat, then it is a flat manifold.*

It is very important for us to know how a LGSSF can be conharmonically flat.

Theorem 12. *A LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is conharmonically flat if and only if $f_2 = 0$ and $(2n + 1)f_1 + 2f_3 = 0$.*

Proof. Comparing (26) with (32), we can get

$$C(U, W)X = \frac{(2n-1)f_1 + 3f_2 + 2f_3}{2n-1} \cdot \{g(W, X)U - g(U, X)W\} + K(U, W)X. \quad (35)$$

If $f_2 = 0$ and $(2n + 1)f_1 + 2f_3 = 0$, then from Theorem 4

$$K(U, W)X = C(U, W)X - \frac{(2n+1)f_1 + 3f_2 + 2f_3}{2n-1} \cdot \{g(W, X)U - g(U, X)W\} = 0. \quad (36)$$

Conversely, if $K(U, W)X = 0$, we know that the conharmonic transformation is a kind of conformal transformation, so if a manifold is conharmonically flat, then it must be conformally flat. In other words, we can get $C(U, W)X = 0$ (equals to $f_2 = 0$) from $K(U, W)X = 0$, that is

$$\begin{aligned} K(U, W)X &= C(U, W)X - \frac{(2n+1)f_1 + 3f_2 + 2f_3}{2n-1} \cdot \{g(W, X)U - g(U, X)W\} \\ &= -\frac{(2n+1)f_1 + 2f_3}{2n-1}\{g(W, X)U - g(U, X)W\} = 0. \end{aligned} \quad (37)$$

We can get $(2n + 1)f_1 + 2f_3 = 0$.

Theorem 13. *A LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is conharmonically flat if and only if $f_2 = 0$ and scalar curvature $r = 0$.*

6. Ricci Semisymmetric Lorentzian Generalized Sasakian-Space-Form

There are many classes of smooth manifolds such as locally symmetric and Ricci symmetric. A smooth manifold is Ricci semisymmetric when the curvature operator $R(U, W)$ acting on S vanishes identically, that is

$$R(U, W) \cdot S = 0. \quad (38)$$

Theorem 14. *A $(2n+1)$-dimensional $(n>1)$ LGSSF $M_1^{2n+1}(f_1, f_2, f_3)$ is Ricci semisymmetric if and only if $f_1+f_3=0$ or $3f_2=(2n-1)f_3$.*

Proof. First, we suppose that $M_1^{2n+1}(f_1, f_2, f_3)$ is Ricci semisymmetric, that is

$$(R(U,W)\cdot S)(Y,Z) = -S(Y, R(U,W)Z) - S(R(U,W)Y, Z) = 0. \tag{39}$$

Put $U=\zeta$ in the above equation, then we will have

$$S(R(\zeta,W)Y,Z) + S(Y, R(\zeta,W)Z) = 0. \tag{40}$$

Then, using (8), we can get

$$\begin{aligned}(f_1+f_3)\{&g(W,Y)S(\zeta,Z)+\eta(Y)S(W,Z)\\ &+g(W,Z)S(\zeta,Y)+\eta(Z)S(W,Y)\}\\ &=(f_1+f_3)((2n-1)f_3-3f_2)\{-2\eta(Y)\eta(W)\eta(Z)\\ &\quad-\eta(Z)g(W,Y)-\eta(Y)g(W,Z)\}=0.\end{aligned} \tag{41}$$

Again we use the orthonormal basis $\{e_1, \cdots, e_{2n+1}=\zeta\}$ with signature $\{\varepsilon_1, \cdots, \varepsilon_{2n}, \varepsilon_{2n+1}=\varepsilon\}$, and this time, in the above equation, we suppose $W=e_i$ and $Z=\varepsilon_i e_i (1\le i\le 2n+1)$, and taking summation over i, we can get

$$2n(f_1+f_3)((2n-1)f_3-3f_2)\eta(Y)=0. \tag{42}$$

Hence, we get $f_1+f_3=0$ or $(2n-1)f_3-3f_2=0$.

Conversely, if $(2n-1)f_3-3f_2=0$, then by direct calculation,

$$\begin{aligned}(R(U,W)\cdot S)(Y,Z) &= -S(Y,R(U,W)Z)-S(R(U,W)Y,Z)\\ &=-(2nf_1+3f_2+f_3)\{g(R(U,W)Z,Y)\\ &\quad+g(R(U,W)Y,Z)\}=0.\end{aligned} \tag{43}$$

If $f_1+f_3=0$, we notice that $\eta(R(U,W)X)=0$, then we will have

$$\begin{aligned}(R(U,W)\cdot S)(Y,Z) &= -S(Y,R(U,W)Z)-S(R(U,W)Y,Z)\\ &=((2n-1)f_3-3f_2)\{g(R(U,W)Z,Y)\\ &\quad+g(R(U,W)Y,Z)\}=0.\end{aligned} \tag{44}$$

Theorem 15. *If a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n>1)$ is conharmonically flat and Ricci semisymmetric, then it is a flat manifold.*

Proof. From Theorem 12 and Theorem 14, we know that if a LGSSF is conharmonically flat and Ricci semisymmetric, then we will have $f_2=0$, $(2n+1)f_1+2f_3=0$, and $3f_2=(2n-1)f_3$ or $f_1+f_3=0$. In any case, we get $f_1=f_2=f_3=0$.

Notice that $f_2=f_3=0$ satisfies $(2n-1)f_3-3f_2=0$, so we can get the following theorem.

Theorem 16. *If a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n>1)$ is projectively flat, then it is Ricci semisymmetric.*

7. Ricci Almost Soliton on Lorentzian Generalized Sasakian-Space-Form

According to [10], we give the definition of the Ricci almost soliton. For a manifold M, if the metric g, along with a vector field W and a function λ defining on M satisfies

$$L_W g + 2S = 2\lambda g, \tag{45}$$

where L_W denotes the Lie derivative, then we call it the triple (g, W, and λ) Ricci almost soliton on the manifold. If $W=\nabla f$ where $f: M \longrightarrow \mathrm{R}$, then we call it the ($g$, ∇f, and λ) gradient Ricci almost soliton. In this case, we call f the potential function and equation (45) will be

$$S + \mathrm{Hess}(f) = \lambda g. \tag{46}$$

According to [17], we have the following definition.

Definition 17. A vector field W on a LGSSF $M_1^{2n+1}(f_1, f_2, f_3)$ is said to be a conformal vector field on the manifold if it satisfies

$$(L_W g)(V,X) = -2\rho g(V,X). \tag{47}$$

ρ is a smooth function on $M_1^{2n+1}(f_1, f_2, f_3)$.

We apply some of our theorems to the Ricci almost soliton and then give two examples to illustrate the application of the following theorem.

Theorem 18. *Let (g, W, and λ) be a Ricci almost soliton on a conformally flat LGSSF $M_1^{2n+1}(f_1, f_2, f_3)(n>1)$. If W is a conformal vector field, then the manifold is projectively flat, so it is Ricci semisymmetric and Einstein.*

Proof. if W is a conformal vector field, we have equation (47). From (45), we get

$$S = (\rho+\lambda)g. \tag{48}$$

Comparing the above equation with (9), we will have the following equations:

$$\begin{aligned}\rho+\lambda &= 2nf_1+3f_2+f_3,\\ 3f_2-(2n-1)f_3 &= 0.\end{aligned} \tag{49}$$

Because $M_1^{2n+1}(f_1, f_2, f_3)$ is conformally flat, we have $f_2=0$ (Theorem 5). Then,

$$f_2=f_3=0. \tag{50}$$

So $M_1^{2n+1}(f_1, f_2, f_3)$ is projectively flat using Theorem 3, and then it is Ricci semisymmetric using Theorem 16.

From Theorem 4, f_1 is a constant function. So

$$S = (\rho + \lambda)g = 2nf_1 g. \tag{51}$$

$M_1^{2n+1}(f_1, f_2, f_3)$ is Einstein.

Example 19. Let $N^{2n}(-2, 0)(n > 1)$ be a generalized complex space-form, then $M_1^{2n+1} = (-\pi/4, \pi/4) \times_h N$ is LGSSF, where

$$h(t) = \sin t + \cos t, \tag{52}$$

and it is conformally flat. The function $f(t, x) = f(t) = a\int_0^t h(s)ds + b, a, b \in \mathbb{R}$ is a potential function. Set $W = -\nabla f$ and $\lambda(t) = -ah'(t) - 2n$, then we have ($g_h$, W, and λ) a gradient Ricci almost soliton on the manifold. M is projectively flat, Einstein and Ricci semisymmetric.

Example 20. In this instance, we consider the generalized complex space-form $N^{2n}(3, 0)(n > 1)$, and the warped product function h is

$$h(t) = \sinh t + 2\cosh t. \tag{53}$$

The warped product $M_1^{2n+1} = \mathbb{R} \times_h N$ is LGSSF and it is conformally flat.

We have (g_h, W, and λ) a gradient Ricci almost soliton on the manifold that $W = -\nabla f$ and $\lambda(t) = -ah'(t) + 2n$, with $f(t, x) = f(t) = a\int_0^t h(s)ds + b, a, b \in \mathbb{R}$. The manifold is projectively flat, Einstein and Ricci semisymmetric.

8. Conclusion

We present the necessary and sufficient conditions for LGSSF to be projectively flat, conformally flat, conharmonically flat, and Ricci semisymmetric. We also study the Ricci almost soliton on LGSSF. As a result, we know how to construct a Lorentzian manifold with certain curvature tensor conditions, which is useful in gauge theories because of the correspondence between curvature and field strength.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interests in this work.

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