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Research Article

An Efficient Compact Difference Method for Temporal Fractional Subdiffusion Equations

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In this paper, a high-order compact finite difference method is proposed for a class of temporal fractional subdiffusion equation. A numerical scheme for the equation has been derived to obtain $2-\alpha$ in time and fourth-order in space. We improve the results by constructing a compact scheme of second-order in time while keeping fourth-order in space. Based on the $L2\text{-}1_\sigma$ approximation formula and a fourth-order compact finite difference approximation, the stability of the constructed scheme and its convergence of second-order in time and fourth-order in space are rigorously proved using a discrete energy analysis method. Applications using two model problems demonstrate the theoretical results.

1. Introduction

The Black-Scholes model, proposed in 1973 by Black and Scholes [1] and Merton [2], gives a theoretical estimate of the price of European-style options. Until Now, some of Black-Scholes models involving the fractional derivatives have emerged. In [3], Wyss priced a European call option by a time-fractional Black-Scholes model. In [4], Liang et al. derive a biparameter fractional Black-Merton-Scholes equation and obtain the explicit option pricing formulas for the European call option and put option, individually. An explicit closed-form analytical solution for barrier options under a generalized time-fractional Black-Scholes model by using eigenfunction expansion method together with the Laplace transform is derived in [5]. In [6], a discrete implicit numerical scheme with a spatially second-order accuracy and a temporally $2-\alpha$ order accuracy is constructed; the stability and convergence of the proposed numerical scheme are analysed using Fourier analysis. In [7], H.Zhang et al. use some numerical technique to price a European double-knock-out barrier option, and then the characteristics of the three fractional Black-Scholes models are analysed through comparison with the classical Black-Scholes model. More recently, a numerical scheme of fourth-order in space and $2-\alpha$ in time is derived in [8]; the solvability and convergence of the proposed numerical scheme are proved rigorously using a Fourier analysis. Some computationally efficient numerical methods have been proposed for solving fractional differential equation, for example, which include finite difference methods, finite element methods, finite volume methods, spectral methods, and meshless methods [9–26].

In this paper, we continue the work of R.H.De Staelen et al. [8]. The class of equations is given by

$$\frac{\partial^\alpha C(S,t)}{\partial t^\alpha} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} + (r-D)S\frac{\partial C(S,t)}{\partial S} = rC(S,t), \quad (S,t) \in (B_d, B_u) \times (0,T) \tag{1}$$

with the following boundary (barrier) and final conditions

$$\begin{aligned} C(B_d,t) &= P(t), \\ C(B_u,t) &= Q(t), \\ & t \in (0,T], \end{aligned} \tag{2}$$

and its initial condition

$$C(S,T) = V(S), \quad S \in [B_d, B_u], \tag{3}$$

where r is the risk free rate, D is the dividend rate, and $\sigma > 0$ is the volatility of the returns. The functions P and

Q are the rebates paid when the corresponding barrier is hit. The terminal playoff of the option is $V(S)$. The fractional derivative in (1) is a Caputo derivative defined as

$$\frac{\partial^\alpha C(S,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial C}{\partial \eta}(S,\eta)(t-\eta)^{-\alpha} \mathrm{d}\eta, & 0<\alpha<1, \\ \frac{\partial C(S,t)}{\partial t}, & \alpha = 1. \end{cases} \tag{4}$$

As described in [8], we consider the transform problem of (1)

$$\begin{aligned} {}_0^C\mathscr{D}_t^\alpha U(x,t) &= a\frac{\partial^2 U(x,t)}{\partial x^2} + b\frac{\partial U(x,t)}{\partial x} - cU(x,t) + f(x,t), \quad (x,t) \in (0,\infty)\times(0,T), \\ U(b_d,t) &= p(t), \\ U(b_u,t) &= q(t), \quad t \in (0,T], \\ U(x,0) &= \varphi(x), \quad x \in [b_d,b_u]. \end{aligned} \tag{5}$$

The rest of the paper is organized as follows: in Section 2, an efficient implicit numerical scheme with second-order accuracy in time and fourth-order accuracy in space is constructed. The analysis of the stability and convergence are presented in Section 3. In Section 4, numerical examples are given to illustrate the accuracy of the presented scheme and to support our theoretical results. Concluding remarks are given in the last section.

2. Construction of the Compact Finite Difference Scheme

In order to simplify the computation and analysis of the following compact finite difference scheme for Black-Scholes model, we use an indirect approach by introducing a suitable transformation.

According to some simple calculations, we transform equation (5) into

$$\begin{aligned} {}_0^C\mathscr{D}_t^\alpha V(x,t) &= a\frac{\partial^2 V(x,t)}{\partial x^2} - dV(x,t) + g(x,t), \quad (x,t) \in (0,\infty)\times(0,T), \\ V(b_d,t) &= p^*(t), \\ V(b_u,t) &= q^*(t), \quad t \in (0,T], \\ V(x,0) &= \varphi^*(x), \quad x \in [b_d,b_u]. \end{aligned} \tag{6}$$

where

$$\begin{aligned} p^*(t) &= p(t), \\ q^*(t) &= k(b_u)\,q(t), \\ \varphi^*(x) &= k(x)\,\varphi(x). \end{aligned} \tag{7}$$

It is clear that $U(x,t)$ is a solution of (5) if and only if $V(x,t)$ is a solution of (6).

In order to construct the compact finite difference scheme for the problem (5), we consider the above equivalent form (6).

Let $\tau = T/N$ be the time step and $h = (b_u - b_d)/M = L/M$ be the spatial step, where M, N are positive integers.

Since the grid function $v = \{v_i \mid 0 \le i \le M\}$, we then define difference operators as follows:

$$\begin{aligned} \delta_x v_{i-1/2} &= \frac{1}{h}(v_i - v_{i-1}), \\ \delta_x^2 v_i &= \frac{1}{h^2}(v_{i+1} - 2v_i + v_{i-1}), \\ \mathscr{H}_x v_i &= v_i + \frac{h^2}{12}\delta_x^2 v_i, \end{aligned} \tag{8}$$

We also define

$$\begin{aligned} a_0 &= \sigma^{1-\alpha}, \\ b_0 &= 0, \\ a_k &= (k+\sigma)^{1-\alpha} - \left(k - \frac{\alpha}{2}\right)^{1-\alpha} \quad (k \ge 1), \\ b_k &= \frac{1}{2-\alpha}\left((k+\sigma)^{2-\alpha} - \left(k-\frac{\alpha}{2}\right)^{2-\alpha}\right) - \frac{1}{2}\left((k+\sigma)^{1-\alpha} + \left(k-\frac{\alpha}{2}\right)^{1-\alpha}\right) \quad (k \ge 1). \end{aligned} \tag{9}$$

where $\sigma = 1 - \alpha/2$, and

$$c_{k,n} = \begin{cases} a_0, & k = 0,\ n = 1 \\ a_k + b_{k+1} - b_k, & 0 \le k \le n-2,\ n \ge 2 \\ a_{n-1} - b_{n-1}, & k = n-1,\ n \ge 2. \end{cases} \tag{10}$$

Lemma 1. *It holds (see [27])*

$$\frac{1-\alpha}{2}(k+\sigma)^{-\alpha} < a_k - b_k < (k+\sigma)^{1-\alpha} - \left(k - \frac{\alpha}{2}\right)^{1-\alpha} \quad (k \ge 1), \tag{11}$$

In order to discretize (6) into a compact finite difference system, we introduce the following lemmas.

Lemma 2. *Assuming $v(t) \in \mathscr{C}^3[0,T]$, we have*

$$ {}_0^C\mathscr{D}_t^\alpha v(t_{n-\alpha/2}) = \frac{1}{\mu}\sum_{k=1}^{n} c_{n-k,n}\left(v(t_k) - v(t_{k-1})\right) + \mathscr{O}\left(\tau^{3-\alpha}\right). \tag{12}$$

where $\mu = \tau^\alpha \Gamma(2-\alpha)$.

Proof. From Lemma 2 of [9], we can obtain the proof of lemma. □

Lemma 3. *Assuming $v(t) \in \mathscr{C}^2[0,T]$. When $n \geq 1$, we obtain*

$$v\left(t_{n-\alpha/2}\right) = \frac{\alpha}{2} v\left(t_{n-1}\right) + \left(1 - \frac{\alpha}{2}\right) v\left(t_n\right) + \mathcal{O}\left(\tau^2\right). \quad (13)$$

Proof. According to some simple calculations, the proof follows from Taylor expansions of the function $v(t)$ at the point $t_{n-\alpha/2}$ for $t = t_{n-1}$ and $t = t_n$. □

Since the above lemmas, we then discretize (6) into a compact finite difference scheme. In order to analyse, we define

$$\begin{aligned} \delta_t v^{n-1/2} &= \frac{1}{\tau}\left(v^n - v^{n-1}\right) \quad (1 \leq n \leq N), \\ v^{n,\alpha/2} &= \frac{\alpha}{2} v^{n-1} + \left(1 - \frac{\alpha}{2}\right) v^n \quad (1 \leq n \leq N). \end{aligned} \quad (14)$$

We also define the grid functions as follows:

$$\begin{aligned} V_i^n &= V\left(x_i, t_n\right), \\ W_i^n &= \frac{\partial V\left(x_i, t_n\right)}{\partial t}, \\ Z_i^n &= \frac{\partial^2 V\left(x_i, t_n\right)}{\partial x^2}, \\ g_i^n &= g\left(x_i, t_n\right), \\ g_i^{n-\alpha/2} &= g\left(x_i, t_{n-\alpha/2}\right), \\ p^{*,n} &= p^*\left(t_n\right), \\ q^{*,n} &= q^*\left(t_n\right), \\ \varphi_i^* &= \varphi^*\left(x_i\right). \end{aligned} \quad (15)$$

For the second-order spatial derivative Z_i^n, we adopt the following fourth-order compact approximation (see [28])

$$\mathscr{H}_x Z_i^n = \delta_x^2 v\left(x_i\right) + O\left(h^4\right), \quad 1 \leq i \leq M-1, \quad (16)$$

We consider equation (6) at the point $(x_i, t_{n-\alpha/2})$; we can obtain

$${}_0^C\mathscr{D}_t^\alpha V\left(x_i, t_{n-\alpha/2}\right) = a Z_i^{n-\alpha/2} - d V_i^{n-\alpha/2} + g_i^{n-\alpha/2}. \quad (17)$$

From Lemmas 2 and 3, we have

$$\begin{aligned} &\frac{1}{\mu}\sum_{k=1}^{n} c_{n-k,n}\left(V_i^k - V_i^{k-1}\right) \\ &\quad = a Z_i^{n,\alpha/2} - d V_i^{n,\alpha/2} + g_i^{n-\alpha/2} + \left(R_t^\alpha\right)_i^n, \\ &\qquad 0 \leq i \leq M, \ 1 \leq n \leq N, \end{aligned} \quad (18)$$

where

$$\begin{aligned} &\left(R_t^\alpha\right)_i^n = (a-d)\,\mathcal{O}\left(\tau^2\right) + \mathcal{O}\left(\tau^{3-\alpha}\right), \\ &\qquad 0 \leq i \leq M, \ 1 \leq n \leq N. \end{aligned} \quad (19)$$

We apply $\mathscr{H}_x$ to equation (18); then we have

$$\begin{aligned} &\frac{1}{\mu}\sum_{k=1}^{n} c_{n-k,n}\mathscr{H}_x\left(V_i^k - V_i^{k-1}\right) \\ &\quad = a\delta_x^2 V_i^{n,\alpha/2} - d\mathscr{H}_x V_i^{n,\alpha/2} + \mathscr{H}_x g_i^{n-\alpha/2} + \left(R_{tx}^\alpha\right)_i^n, \\ &\qquad 1 \leq i \leq M-1, \ 1 \leq n \leq N, \end{aligned} \quad (20)$$

where

$$\begin{aligned} &\left(R_{tx}^\alpha\right)_i^n = \mathscr{H}_x\left(R_t^\alpha\right)_i^n + a\left(R_x^\alpha\right)_i^n, \\ &\qquad 1 \leq i \leq M-1, \ 1 \leq n \leq N \end{aligned} \quad (21)$$

and

$$\begin{aligned} &\left|\left(R_{tx}^\alpha\right)_i^n\right| \leq C_R\left(\tau^2 + h^4\right), \\ &\qquad 1 \leq i \leq M-1, \ 1 \leq n \leq N. \end{aligned} \quad (22)$$

If we omit $(R_{tx}^\alpha)_i^n$, then we have the compact finite difference scheme:

$$\begin{aligned} &\frac{1}{\mu}\sum_{k=1}^{n} c_{n-k,n}\mathscr{H}_x\left(v_i^k - v_i^{k-1}\right) \\ &\quad = a\delta_x^2 v_i^{n,\alpha/2} - d\mathscr{H}_x v_i^{n,\alpha/2} + \mathscr{H}_x g_i^{n-\alpha/2}, \\ &\qquad 1 \leq i \leq M-1, \ 1 \leq n \leq N, \\ &v\left(b_d, t\right) = p^*(t), \\ &v\left(b_u, t\right) = q^*(t), \\ &\qquad t \in (0, T], \\ &v(x, 0) = \varphi^*(x), \quad x \in \left[b_d, b_u\right]. \end{aligned} \quad (23)$$

3. Stability and Convergence of the Proposed Compact Difference Scheme

Theorem 4. *The compact difference scheme (23) is uniquely solvable.*

Proof. The compact difference scheme (23) can be written in matrix form

$$\mathscr{A}\mathbf{V}^n = \mathbf{b}_{n-1}, \quad (24)$$

where

$$\mathbf{b}_{n-1} = \sum_{k=0}^{n-1}\zeta_k \mathbf{V}^k, \quad \zeta_k \in \mathbf{R} \quad (25)$$

The tridiagonal coefficient matrix $\mathscr{A} = (a_{ij})$ yields

$$\begin{aligned} \left|a_{ii}\right| &= \frac{5}{6}\left(\frac{\left(a_0^{(\alpha)} + b_1^{(\alpha)}\right)}{\mu}\right) + \frac{a(2-\alpha)}{h^2} + \frac{5(2-\alpha)}{12}d, \\ \sum_{j\neq i}\left|a_{ij}\right| &= \left|\frac{1}{6}\left(\frac{\left(a_0^{(\alpha)} + b_1^{(\alpha)}\right)}{\mu}\right) - \frac{a(2-\alpha)}{h^2} + \frac{(2-\alpha)}{12}d\right|. \end{aligned} \quad (26)$$

It is easy to see that the tridiagonal coefficient matrix $\mathscr{A}$ is strictly diagonally dominant. Therefore, the coefficient matrix is nonsingular and hence invertible. □

Next, we consider the stability and convergence analysis of the compact difference scheme (23).

Letting $\Omega = \{u \mid u = (u_0, u_1, \ldots, u_M), u_0 = u_M = 0\}$, for grid functions $u, v \in \Omega$, we define the inner product and norm as follows:

$$\begin{gathered}(u, v) = h \sum_{i=1}^{M-1} u_i v_i, \\ \|u\| = (u, u)^{1/2}, \\ \|u\|_\infty = \max_{0 \le i \le M} |u_i|. \\ (\delta_x u, \delta_x v) = h \sum_{i=0}^{M-1} \delta_x u_{i+1/2} \delta_x v_{i+1/2}, \\ |u|_1 = (\delta_x u, \delta_x u]^{1/2}, \\ \|u\|_1 = \left(\|u\|^2 + |u|_1^2\right)^{1/2}.\end{gathered} \tag{27}$$

According to simple calculations, we obtain

$$\begin{gathered}\left(\delta_x^2 u, v\right) = -(\delta_x u, \delta_x v), \\ h\left\|\delta_x^2 u\right\| \le 2|u|_1, \\ h|u|_1 \le 2\|u\|.\end{gathered} \tag{28}$$

In order to analyse, we introduce the discrete inner product and norm:

$$\begin{gathered}\langle u, v\rangle = \left(\mathscr{H}_x u, -\delta_x^2 v\right) = (\delta_x u, \delta_x v] - \frac{h^2}{12}\left(\delta_x^2 u, \delta_x^2 v\right), \\ \|u\|_\varepsilon = \langle u, u\rangle^{1/2}.\end{gathered} \tag{29}$$

Based on above inner product and norm, we have the following lemmas.

Lemma 5 (see [29]). *Suppose $u \in \Omega$, we obtain*

$$\begin{gathered}\left\|\mathscr{H}_x u\right\|^2 \le \|u\|^2 \le \frac{3L^2}{16}\|u\|_\varepsilon^2, \\ \|u\|_\infty^2 \le \frac{3L}{8}\|u\|_\varepsilon^2, \\ \|u\|_1^2 \le \frac{3\left(8 + L^2\right)}{16}\|u\|_\varepsilon^2.\end{gathered} \tag{30}$$

Lemma 6 (see [27]). *Suppose $u \in \Omega$, we obtain*

$$\begin{gathered}\|u\| \le \frac{L^2}{8}\left\|\delta_x^2 u\right\|, \\ \|u\|_\varepsilon^2 \le \frac{3L^2}{16}\left\|\delta_x^2 u\right\|^2.\end{gathered} \tag{31}$$

Lemma 7 (see [9]). *Suppose $u \in \Omega$, we obtain*

$$\begin{aligned}&\left(\sum_{k=1}^{n} c_{n-k,n}\mathscr{H}_x\left(u^k - u^{k-1}\right), -\delta_x^2 u^{n,\alpha/2}\right) \\ &\quad\ge \frac{1}{2}\sum_{k=1}^{n} c_{n-k,n}\left(\left\|u^k\right\|_\varepsilon^2 - \left\|u^{k-1}\right\|_\varepsilon^2\right), \quad 1 \le n \le N.\end{aligned} \tag{32}$$

In the next, we then analyse the stability and convergence of the scheme (23).

Theorem 8 (stability). *Let $v^n = (v_0^n, v_1^n, \ldots, v_M^n)$ be the solution of the compact difference scheme (23) with $v_0^n = v_M^n = 0$. Assume that one of the conditions $1 \le 4(4\varepsilon - 1)a/3d\varepsilon L^2$ holds for some positive constant $\varepsilon > 1/4$.*

Then it holds

$$\left\|v^n\right\|_\varepsilon^2 \le \left\|v^0\right\|_\varepsilon^2 + \frac{4\varepsilon\Gamma(1-\alpha)T^\alpha}{a}\max_{1 \le n \le N}\left\|\mathscr{H}_x g^{n-\alpha/2}\right\|^2, \quad 1 \le n \le N. \tag{33}$$

Proof. We take the inner product of equation (23) with $-\delta_x^2 v^{n,\alpha/2}$ yield

$$\begin{aligned}&\frac{1}{\mu}\left(\sum_{k=1}^{n} c_{n-k,n}\mathscr{H}_x\left(v^k - v^{k-1}\right), -\delta_x^2 v^{n,\alpha/2}\right) \\ &\quad= -a\left\|\delta_x^2 v^{n,\alpha/2}\right\|^2 - d\left(\mathscr{H}_x\left(v^{n,\alpha/2}\right), \delta_x^2 v^{n,\alpha/2}\right) \\ &\quad\quad- \left(\mathscr{H}_x g^{n-\alpha/2}, \delta_x^2 v^{n,\alpha/2}\right), \quad 1 \le n \le N.\end{aligned} \tag{34}$$

Using Lemma 7,

$$\begin{aligned}&\frac{1}{2\mu}\sum_{k=1}^{n} c_{n-k,n}\left(\left\|v^k\right\|_\varepsilon^2 - \left\|v^{k-1}\right\|_\varepsilon^2\right) \\ &\quad\le -a\left\|\delta_x^2 v^{n,\alpha/2}\right\|^2 + d\left\|v^{n,\alpha/2}\right\|_\varepsilon^2 \\ &\quad\quad- \left(\mathscr{H}_x g^{n-\alpha/2}, \delta_x^2 u^{n,\alpha/2}\right), \quad 1 \le n \le N.\end{aligned} \tag{35}$$

When $1 \le 4(4\varepsilon - 1)a/3d\varepsilon L^2$ for some positive constant $\varepsilon > 1/4$, we have from the Cauchy-Schwarz inequality and Lemmas 6 that

$$\begin{aligned}d\left\|v^{n,\alpha/2}\right\|_\varepsilon^2 &\le \frac{3dL^2}{16}\left\|\delta_x^2 v^{n,\alpha/2}\right\|^2 \\ &\le \left(a - \frac{a}{4\varepsilon}\right)\left\|\delta_x^2 v^{n,\alpha/2}\right\|^2\end{aligned} \tag{36}$$

$$\begin{aligned}&-\left(\mathscr{H}_x g^{n-\alpha/2}, \delta_x^2 u^{n,\alpha/2}\right) \\ &\quad\le \frac{\varepsilon}{a}\left\|\mathscr{H}_x g^{n-\alpha/2}\right\|^2 + \frac{a}{4\varepsilon}\left\|\delta_x^2 v^{n,\alpha/2}\right\|^2\end{aligned} \tag{37}$$

By (35) and the Cauchy-Schwarz inequality,

$$\begin{aligned}&-a\left\|\delta_x^2 u^{n,\alpha/2}\right\|^2 + d\left\|v^{n,\alpha/2}\right\|_\varepsilon^2 - \left(\mathscr{H}_x g^{n-\alpha/2}, \delta_x^2 u^{n,\alpha/2}\right) \\ &\quad\le \frac{\varepsilon}{a}\left\|\mathscr{H}_x g^{n-\alpha/2}\right\|^2.\end{aligned} \tag{38}$$

Substituting (38) into (35) leads to

$$\sum_{k=1}^{n} c_{n-k,n}\left(\left\|u^{k}\right\|_{\varepsilon}^{2}-\left\|u^{k-1}\right\|_{\varepsilon}^{2}\right) \leq \frac{2\varepsilon\mu}{a}\left\|\mathscr{H}_{x} g^{n-\alpha/2}\right\|^{2}. \quad (39)$$

The above inequality can be rewritten as

$$\begin{aligned} c_{0,n}\left\|u^{n}\right\|_{\varepsilon}^{2} \leq & \sum_{k=1}^{n-1}\left(c_{n-k-1,n}-c_{n-k,n}\right)\left\|u^{k}\right\|_{\varepsilon}^{2}+c_{n-1,n}\left\|u^{0}\right\|_{*}^{2} \\ & +\frac{2\varepsilon\mu}{a}\left\|\mathscr{H}_{x} g^{n-\alpha/2}\right\|^{2}. \end{aligned} \quad (40)$$

Since by the definition of $c_{n-1,n}$,

$$\frac{\mu}{c_{n-1,n}}=\frac{\mu}{a_{n-1}-b_{n-1}}<2\Gamma(1-\alpha) T^{\alpha}, \quad (41)$$

we have from (40) that

$$\begin{aligned} & c_{0,n}\left\|u^{n}\right\|_{\varepsilon}^{2} \\ & \quad \leq \sum_{k=1}^{n-1}\left(c_{n-k-1,n}-c_{n-k,n}\right)\left\|u^{k}\right\|_{\varepsilon}^{2} \\ & \quad + c_{n-1,n}\left(\left\|u^{0}\right\|_{\varepsilon}^{2}+\frac{4\varepsilon\Gamma(1-\alpha) T^{\alpha}}{a}\left\|\mathscr{H}_{x} g^{n-\alpha/2}\right\|^{2}\right). \end{aligned} \quad (42)$$

Letting

$$E=\left\|u^{0}\right\|_{\varepsilon}^{2}+\frac{4\varepsilon\Gamma(1-\alpha) T^{\alpha}}{a}\max_{1\leq n\leq N}\left\|\mathscr{H}_{x} g^{n-\alpha/2}\right\|^{2} \quad (43)$$

and assuming $\|u^{k}\|_{\varepsilon}^{2} \leq E(0 \leq k \leq n-1)$, we obtain

$$c_{0,n}\left\|u^{n}\right\|_{\varepsilon}^{2} \leq \sum_{k=1}^{n-1}\left(c_{n-k-1,n}-c_{n-k,n}\right) E+c_{n-1,n} E=c_{0,n} E. \quad (44)$$

and we have the needed estimates. □

Letting $e_i^n = V_i^n - v_i^n$, we get the following error equation:

$$\begin{aligned} & \frac{1}{\mu}\sum_{k=1}^{n} c_{n-k,n}\mathscr{H}_{x}\left(e_{i}^{k}-e_{i}^{k-1}\right) \\ & \quad = a\delta_{x}^{2} e_{i}^{n,\alpha/2}-d\mathscr{H}_{x} e_{i}^{n,\alpha/2}+\left(R_{tx}^{\alpha}\right)_{i}^{n}, \\ & \qquad 1 \leq i \leq M-1, \ 1 \leq n \leq N, \\ & e\left(b_{d}, t\right)=0, \\ & e\left(b_{u}, t\right)=0, \\ & \qquad t \in (0, T], \\ & e(x, 0)=0, \quad x \in\left[b_{d}, b_{u}\right]. \end{aligned} \quad (45)$$

Since the above error equation (45), we now obtain the following convergence results.

Theorem 9 (convergence). *Let V_i^n denote the value of the solution $v(x,t)$ of (23) at the mesh point (x_i, t_n) and let $v^n = (v_0^n, v_1^n, \ldots, v_M^n)$ be the solution of the compact difference scheme (23). Then when $1 \leq 4(4\varepsilon-1)a/3d\varepsilon L^2$, it holds*

$$\left\|U^{n}-u^{n}\right\|_{\varepsilon} \leq C_{1}\left(\tau^{2}+h^{4}\right), \quad 1 \leq n \leq N, \quad (46)$$

where

$$C_{1}=\left(\frac{4\Gamma(1-\alpha) T^{\alpha} L C_{R}^{2}}{a}\right)^{1/2}, \quad (47)$$

Proof. It follows from Theorem 8 that

$$\left\|e^{n}\right\|_{\varepsilon}^{2} \leq \frac{4\varepsilon\Gamma(1-\alpha) T^{\alpha}}{a}\max_{1\leq n\leq N}\left\|\left(R_{tx}^{\alpha}\right)_{i}^{n}\right\|^{2}, \quad 1 \leq n \leq N, \quad (48)$$

Applying (22), we get

$$\left\|e^{n}\right\|_{\varepsilon}^{2} \leq C_{1}^{2}\left(\tau^{2}+h^{4}\right)^{2}. \quad (49)$$

The estimate (46) is proved. □

Remark 10. The constraint condition $1 \leq 4(4\varepsilon-1)a/3d\varepsilon L^2$ in Theorems 8 and 9 is only for the analysis of the stability and convergence of the compact difference scheme (23). This condition is easily verifiable for practical problems.

4. Numerical Experiment

For demonstrating the efficiency of the compact difference scheme (23), we make two numerical experiments of it.

Suppose $V_i^n = v(x_i, t_n)$ be the value of the solution $v(x,t)$ of the problem (1)–(3) at the mesh point (x_i, t_n). From (22), we can see that

$$\left\|V^{n}-v^{n}\right\|_{\nu} \leq C_{2}\left(\tau^{2}+h^{4}\right), \quad \nu=1,2,\infty \quad (50)$$

where C_2 is a positive constant independent. In order to check this accuracy of the compact difference scheme, we compute the following norm errors:

$$\mathrm{E}_{\nu}(\tau, h)=\max_{0\leq n\leq N}\left\|V^{n}-v^{n}\right\|_{\nu} \quad (\nu=1,2,\infty). \quad (51)$$

The temporal convergence order and the spatial convergence order are denoted by

$$\begin{aligned} & \mathrm{O}_{\nu}^{\mathrm{t}}(\tau, h)=\log_{2}\left(\frac{\mathrm{E}_{\nu}(2\tau, h)}{\mathrm{E}_{\nu}(\tau, h)}\right), \\ & \mathrm{O}_{\nu}^{\mathrm{s}}(\tau, h)=\log_{2}\left(\frac{\mathrm{E}_{\nu}(\tau, 2h)}{\mathrm{E}_{\nu}(\tau, h)}\right) \\ & \qquad (\nu=1,2,\infty). \end{aligned} \quad (52)$$

Example 1. We first consider a problem, which is governed by equation (1) in $[0,1]\times[0,1]$ with $r = 0.05, \sigma = 0.25, a = \sigma^2/2, b = r - a, c = r$ and

$$\begin{aligned} f(x, t)= & \left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}+\frac{2t^{1-\alpha}}{\Gamma(2-\alpha)} x^{2}(1-x)-(t+1)^{2}\right. \\ & \left.\cdot\left[a(2-6x)+b\left(2x-3x^{2}\right)-cx^{2}(1-x)\right]\right). \end{aligned} \quad (53)$$

Table 1: The errors and the temporal convergence orders of the compact difference scheme (23) for Example 1 ($h = 1/100$).

α	τ	$\mathrm{E}_1(\tau,h)$	$\mathrm{O}_1^{\mathrm{t}}(\tau,h)$	$\mathrm{E}_2(\tau,h)$	$\mathrm{O}_2^{\mathrm{t}}(\tau,h)$	$\mathrm{E}_\infty(\tau,h)$	$\mathrm{O}_\infty^{\mathrm{t}}(\tau,h)$
1/4	1/10	3.7361e−05		3.7314e−05		6.2608e−05	
	1/20	9.3611e−06	1.9968	9.3492e−06	1.9968	1.5684e−05	1.9970
	1/40	2.3429e−06	1.9984	2.3399e−06	1.9984	3.9251e−06	1.9985
	1/80	5.8603e−07	1.9992	5.8528e−07	1.9992	9.8176e−07	1.9993
	1/160	1.4654e−07	1.9997	1.4635e−07	1.9997	2.4549e−07	1.9997
	1/320	3.6634e−08	2.0001	3.6587e−08	2.0001	6.1372e−08	2.0000
1/2	1/10	6.7788e−05		6.7702e−05		1.1393e−04	
	1/20	1.6994e−05	1.9960	1.6972e−05	1.9960	2.8555e−05	1.9964
	1/40	4.2543e−06	1.9980	4.2489e−06	1.9980	7.1480e−06	1.9981
	1/80	1.0643e−06	1.9990	1.0630e−06	1.9990	1.7882e−06	1.9991
	1/160	2.6617e−07	1.9995	2.6583e−07	1.9995	4.4718e−07	1.9996
	1/320	6.6548e−08	1.9999	6.6463e−08	1.9999	1.1181e−07	1.9999
3/4	1/10	8.8226e−05		8.8110e−05		1.4950e−04	
	1/20	2.2098e−05	1.9973	2.2069e−05	1.9973	3.7435e−05	1.9976
	1/40	5.5299e−06	1.9986	5.5226e−06	1.9986	9.3672e−06	1.9987
	1/80	1.3832e−06	1.9993	1.3813e−06	1.9993	2.3429e−06	1.9993
	1/160	3.4587e−07	1.9996	3.4542e−07	1.9996	5.8585e−07	1.9997
	1/320	8.6475e−08	1.9999	8.6361e−08	1.9999	1.4647e−07	1.9999

Table 2: The errors and the spatial convergence orders of the compact difference scheme (23) for Example 1 ($h = 1/10000$).

α	τ	$\mathrm{E}_1(\tau,h)$	$\mathrm{O}_1^{\mathrm{t}}(\tau,h)$	$\mathrm{E}_2(\tau,h)$	$\mathrm{O}_2^{\mathrm{t}}(\tau,h)$	$\mathrm{E}_\infty(\tau,h)$	$\mathrm{O}_\infty^{\mathrm{t}}(\tau,h)$
1/4	1/2	1.1190e−04		6.4607e−05		9.1369e−05	
	1/4	5.4155e−06	4.3690	4.2667e−06	3.9205	5.4429e−06	4.0693
	1/8	2.9041e−07	4.2209	2.6840e−07	3.9907	3.3922e−07	4.0041
	1/16	1.7125e−08	4.0840	1.6758e−08	4.0015	2.1165e−08	4.0024
	1/32	1.0264e−09	4.0604	1.0207e−09	4.0372	1.2889e−09	4.0375
1/2	1/2	1.0340e−04		5.9701e−05		8.4430e−05	
	1/4	5.0151e−06	4.3659	3.9472e−06	3.9189	4.9995e−06	4.0779
	1/8	2.6907e−07	4.2202	2.4841e−07	3.9900	3.1142e−07	4.0048
	1/16	1.5831e−08	4.0871	1.5487e−08	4.0036	1.9400e−08	4.0047
	1/32	9.2534e−10	4.0967	9.2011e−10	4.0731	1.1538e−09	4.0716
3/4	1/2	9.3459e−05		5.3959e−05		7.6309e−05	
	1/4	4.5477e−06	4.3611	3.5734e−06	3.9165	4.4770e−06	4.0912
	1/8	2.4420e−07	4.2190	2.2506e−07	3.9889	2.7865e−07	4.0060
	1/16	1.4335e−08	4.0905	1.4014e−08	4.0053	1.7372e−08	4.0036
	1/32	8.2043e−10	4.1270	8.1562e−10	4.1028	1.0150e−09	4.0971

The boundary and initial conditions are given by (2) and (3) with

$$U(x,0) = x^2(1-x),$$
$$U(0,t) = U(1,t) = 0. \tag{54}$$

It is easy to check that $U(x,t) = (t+1)^2 x^2 (1-x)$ is the solution of this problem.

For different α, we let the spatial step $h = 1/100$. Table 1 gives the errors $\mathrm{E}_\nu(\tau,h)$ $(\nu = 1,2,\infty)$ and the temporal convergence orders $\mathrm{O}_\nu^{\mathrm{t}}(\tau,h)$ $(\nu = 1,2,\infty)$ of the computed solution U_i^n for $\alpha = 1/4, 1/2, 3/4$ and different time step τ. From the table, we can see that the computed solution U_i^n has the second-order temporal accuracy. For comparison, the corresponding temporal convergence orders $\mathrm{O}_\nu^{\mathrm{t}}(\tau,h)$ $(\nu = \infty)$ given in [8] has only $2-\alpha$ order; thus it is far less accurate than the compact difference scheme (23) given in this paper.

Next, we compute the spatial convergence order of the compact difference scheme (23). Table 2 presents the errors $\mathrm{E}_\nu(\tau,h)$ $(\nu = 1,2,\infty)$ and the spatial convergence orders $order\mathrm{O}_\nu^{\mathrm{s}}(\tau,h)$ $(\nu = 1,2,\infty)$. The table demonstrates that the compact difference scheme (23) has the fourth-order spatial accuracy.

Example 2. In this example, we test the error and the convergence order of the compact difference scheme (23).

Table 3: The errors and the temporal convergence orders of the compact difference scheme (23) for Example 2 ($\tau = 1/100$).

α	h	$E_1(\tau,h)$	$O_1^s(\tau,h)$	$E_2(\tau,h)$	$O_2^s(\tau,h)$	$E_\infty(\tau,h)$	$O_\infty^s(\tau,h)$
1/4	1/10	9.5681e−05		9.5622e−05		1.3266e−04	
	1/20	2.3985e−05	1.9961	2.3971e−05	1.9961	3.3254e−05	1.9961
	1/40	6.0044e−06	1.9981	6.0007e−06	1.9981	8.3244e−06	1.9981
	1/80	1.5021e−06	1.9990	1.5012e−06	1.9990	2.0825e−06	1.9991
	1/160	3.7564e−07	1.9996	3.7541e−07	1.9996	5.2078e−07	1.9996
	1/320	9.3916e−08	1.9999	9.3859e−08	1.9999	1.3020e−07	1.9999
1/2	1/10	1.7283e−04		1.7272e−04		2.4033e−04	
	1/20	4.3358e−05	1.9950	4.3331e−05	1.9950	6.0286e−05	1.9951
	1/40	1.0858e−05	1.9975	1.0852e−05	1.9975	1.5097e−05	1.9976
	1/80	2.7169e−06	1.9987	2.7153e−06	1.9987	3.7774e−06	1.9988
	1/160	6.7952e−07	1.9994	6.7910e−07	1.9994	9.4474e−07	1.9994
	1/320	1.6991e−07	1.9998	1.6980e−07	1.9998	2.3622e−07	1.9998
3/4	1/10	2.2075e−04		2.2061e−04		3.0894e−04	
	1/20	5.5326e−05	1.9964	5.5291e−05	1.9964	7.7418e−05	1.9966
	1/40	1.3849e−05	1.9981	1.3841e−05	1.9981	1.9378e−05	1.9983
	1/80	3.4646e−06	1.9991	3.4624e−06	1.9991	4.8476e−06	1.9991
	1/160	8.6641e−07	1.9995	8.6587e−07	1.9995	1.2123e−06	1.9996
	1/320	2.1663e−07	1.9998	2.1650e−07	1.9998	3.0311e−07	1.9998

Table 4: The errors and the spatial convergence orders of the compact difference scheme (23) for Example 2 ($h = 1/15000$).

α	τ	$E_1(\tau,h)$	$O_1^t(\tau,h)$	$E_2(\tau,h)$	$O_2^t(\tau,h)$	$E_\infty(\tau,h)$	$O_\infty^t(\tau,h)$
1/4	1/2	1.2909e−04		7.4532e−05		1.0540e−04	
	1/4	6.2469e−06	4.3691	4.9219e−06	3.9206	6.2789e−06	4.0693
	1/8	3.3499e−07	4.2210	3.0961e−07	3.9907	3.9130e−07	4.0042
	1/16	1.9745e−08	4.0846	1.9323e−08	4.0021	2.4403e−08	4.0031
	1/32	1.1750e−09	4.0707	1.1685e−09	4.0475	1.4745e−09	4.0488
1/2	1/2	1.1929e−04		6.8871e−05		9.7399e−05	
	1/4	5.7849e−06	4.3660	4.5533e−06	3.9189	5.7675e−06	4.0779
	1/8	3.1036e−07	4.2203	2.8654e−07	3.9901	3.5923e−07	4.0050
	1/16	1.8246e−08	4.0883	1.7849e−08	4.0048	2.2357e−08	4.0061
	1/32	1.0507e−09	4.1182	1.0447e−09	4.0946	1.3056e−09	4.0980
3/4	1/2	1.0782e−04		6.2247e−05		8.8031e−05	
	1/4	5.2457e−06	4.3613	4.1220e−06	3.9166	5.1648e−06	4.0912
	1/8	2.8166e−07	4.2191	2.5959e−07	3.9891	3.2142e−07	4.0062
	1/16	1.6515e−08	4.0921	1.6146e−08	4.0070	2.0016e−08	4.0053
	1/32	9.2416e−10	4.1595	9.1874e−10	4.1354	1.1316e−09	4.1447

Consider equation (1) in the domain $[0,1] \times [0,1]$ with $r = 0.5, a = 1, b = r - a, c = r$ and

$$f(x,t) = \left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{2t^{1-\alpha}}{\Gamma(2-\alpha)} \left(x^3 + x^2 + 1\right) - (t+1)^2 \cdot \left[a(2+6x) + b\left(2x + 3x^2\right) - c\left(x^3 + x^2 + 1\right)\right] \right). \tag{55}$$

The boundary and initial conditions are given by (2) and (3) with

$$\begin{aligned} \phi_0(t) &= x^3 + x^2 + 1, \\ U(0,t) &= (t+1)^2, \\ U(1,t) &= 3(t+1)^2. \end{aligned} \tag{56}$$

It is clear that $U(x,t) = (t+1)^2(x^3 + x^2 + 1)$ is the exact analytical solution of this problem.

Apply the compact difference scheme (23) to solve the above problem. Table 3 presents the errors $E_\nu(\tau,h)$ ($\nu = 1, 2, \infty$) and the temporal convergence orders $O_\nu^t(\tau,h)$ ($\nu = 1, 2, \infty$); we can see that the computed solution U_i^n has the second-order temporal accuracy.

From Table 4, we can obtain the errors $E_\nu(\tau,h)$ ($\nu = 1, 2, \infty$) and the spatial convergence orders $O_\nu^s(\tau,h)$ ($\nu = 1, 2, \infty$).

These numerical results demonstrate that the accuracy of the compact difference scheme (23) is fourth-order.

5. Concluding Remarks

In this paper, a high-order compact finite difference method for a class of time-fractional Black-Scholes equations is presented and analysed. We apply the $L2\text{-}1_\sigma$ approximation formula to the Caputo derivative; then we construct a fourth-order compact finite difference approximation for the spatial derivative. We have analysed the solvability, stability, and convergence of the constructed scheme and provided the optimal error estimates. The constructed scheme has the second-order temporal accuracy and the fourth-order spatial accuracy, which improves the temporal accuracy of the method given in [8].

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] F. Black and M. Scholes, "The pricing of options corporate liabilities," *Journal of Political Economy*, vol. 81, pp. 637–659, 1973.

[2] R. C. Merton, "Theory of rational option pricing," *Bell Journal of Economics and Management Science*, vol. 4, pp. 141–183, 1973.

[3] W. Wyss, "The fractional Black-Scholes equation," *Fractional Calculus and Applied Analysis*, vol. 3, no. 1, pp. 51–61, 2000.

[4] J.-R. Liang, J. Wang, W.-J. Zhang, W.-Y. Qiu, and F.-Y. Ren, "Option pricing of a bi-fractional Black-Merton-Scholes model with the Hurst exponent H in [1/2,1]," *Applied Mathematics Letters*, vol. 23, no. 8, pp. 859–863, 2010.

[5] W. Chen, X. Xu, and S.-P. Zhu, "Analytically pricing double barrier options based on a time-fractional Black-Scholes equation," *Computers & Mathematics with Applications*, vol. 69, no. 12, pp. 1407–1419, 2015.

[6] H. Zhang, F. Liu, I. Turner, and Q. Yang, "Numerical solution of the time fractional Black-Scholes model governing European options," *Computers & Mathematics with Applications*, vol. 71, no. 9, pp. 1772–1783, 2016.

[7] H. Zhang, F. Liu, I. Turner, and S. Chen, "The numerical simulation of the tempered fractional Black–Scholes equation for European double barrier option," *Applied Mathematical Modelling*, vol. 40, no. 11-12, pp. 5819–5834, 2016.

[8] R. De Staelen and A. Hendy, "Numerically pricing double barrier options in a time-fractional Black–Scholes model," *Computers & Mathematics with Applications*, vol. 74, no. 6, pp. 1166–1175, 2017.

[9] A. A. Alikhanov, "A new difference scheme for the time fractional diffusion equation," *Journal of Computational Physics*, vol. 280, pp. 424–438, 2015.

[10] A. Bueno-Orovio, D. Kay, V. Grau, B. Rodriguez, and K. Burrage, "Fractional diffusion models of cardiac electrical propagation: role of structural heterogeneity in dispersion of repolarization," *Journal of the Royal Society Interface*, vol. 11, article no. 352, pp. 20140352-20140352, 2014.

[11] K. Burrage, N. Hale, and D. Kay, "An efficient implicit FEM scheme for fractional-in-space reaction-diffusion equations," *SIAM Journal on Scientific Computing*, vol. 34, no. 4, pp. A2145–A2172, 2012.

[12] S. Chen, F. Liu, P. Zhuang, and V. Anh, "Finite difference approximations for the fractional Fokker-Planck equation," *Applied Mathematical Modelling: Simulation and Computation for Engineering and Environmental Systems*, vol. 33, no. 1, pp. 256–273, 2009.

[13] Y. Dimitrov, "Numerical approximations for fractional differential equations," *Journal of Fractional Calculus and Applications*, vol. 5, pp. 1–45, 2014.

[14] G.-h. Gao and Z.-z. Sun, "A compact finite difference scheme for the fractional sub-diffusion equations," *Journal of Computational Physics*, vol. 230, no. 3, pp. 586–595, 2011.

[15] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, NY, USA, 1999.

[16] X. Li and C. Xu, "A space-time spectral method for the time fractional diffusion equation," *SIAM Journal on Numerical Analysis*, vol. 47, no. 3, pp. 2108–2131, 2009.

[17] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[18] T. A. M. Langlands and B. I. Henry, "The accuracy and stability of an implicit solution method for the fractional diffusion equation," *Journal of Computational Physics*, vol. 205, no. 2, pp. 719–736, 2005.

[19] Y. Luchko and A. Punzi, "Modeling anomalous heat transport in geothermal reservoirs via fractional diffusion equations," *GEM - International Journal on Geomathematics*, vol. 1, no. 2, pp. 257–276, 2011.

[20] C. Li and F. Zeng, *Numerical Methods for Fractional Calculus*, Chapman and Hall/CRC, Boca Raton, FL, USA, 2015.

[21] R. Metzler and J. Klafter, "The random walk's guide to anomalous diffusion: a fractional dynamics approach," *Physics Reports*, vol. 339, no. 1, pp. 1–77, 2000.

[22] A. A. Samarskii, *The Theory of Difference Schemes*, Marcel Dekker, New York, NY, USA, 2001.

[23] Z. Wang and S. Vong, "Compact difference schemes for the modified anomalous fractional sub-diffusion equation and the fractional diffusion-wave equation," *Journal of Computational Physics*, vol. 277, pp. 1–15, 2014.

[24] Y. Zhang, D. A. Benson, and D. M. Reeves, "Time and space nonlocalities underlying fractional-derivative models: distinction and literature review of field applications," *Advances in Water Resources*, vol. 32, no. 4, pp. 561–581, 2009.

[25] L. Zhao and W. Deng, "A series of high-order quasi-compact schemes for space fractional diffusion equations based on the superconvergent approximations for fractional derivatives," *Numerical Methods for Partial Differential Equations*, vol. 31, no. 5, pp. 1345–1381, 2015.

[26] Y. Guo, "Solvability for a nonlinear fractional differential equation," *Bulletin of the Australian Mathematical Society*, vol. 80, no. 1, pp. 125–138, 2009.

[27] Y. Wang and L. Ren, "Efficient compact finite difference methods for a class of time-fractional convection–reaction–diffusion equations with variable coefficients," *International Journal of Computer Mathematics*, vol. 96, no. 2, pp. 264–297, 2018.

[28] Y. N. Zhang, Z. Z. Sun, and H. W. Wu, "Error estimates of Crank-Nicolson-type difference schemes for the subdiffusion equation," *SIAM Journal on Numerical Analysis*, vol. 49, pp. 2302–2322, 2011.

[29] Y. Wang, "A compact finite difference method for solving a class of time fractional convection-subdiffusion equations," *BIT Numerical Mathematics*, vol. 55, no. 4, pp. 1187–1217, 2015.