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Research Article

Fuzzy Optimal Control Problem of Several Variables

Altyeb Mohammed Mustafa,[1,2] **Zengtai Gong**,[1] **and Mawia Osman**[1]

[1]*College of Mathematics and Statistics, Northwest Normal University, Lanzhou, Gansu, China*
[2]*Department of Applied Mathematics, Faculty of Mathematical Science, University of Khartoum, Khartoum, Sudan*

Correspondence should be addressed to Zengtai Gong; zt-gong@163.com

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The purpose of this paper is to establish the necessary conditions for a fuzzy optimal control problem of several variables. Also, we define fuzzy optimal control problems involving isoperimetric constraints and higher order differential equations. Then, we convert these problems to fuzzy optimal control problems of several variables in order to solve these problems using the same solution method. The main results of this paper are illustrated throughout three examples, more specifically, a discussion on the strong solutions (fuzzy solutions) of our problems.

1. Introduction

Optimal control theory is considered as a modern extension of the classical calculus of variations; however, it differs from calculus of variations in that it uses control variables to optimize the function. The development of the mathematical theory for optimal control began in the early 1950's, partially in response to problems in various branches of engineering and economics. The study of classical optimal control theory from different viewpoints greatly attracted the attention of many mathematicians, and the detailed arguments can be found in many textbooks, for instance, [1], and references therein. Moreover, optimal control strategy, i.e., solving necessary conditions for optimality, can be applied in several fields, such as economy, biology, and process engineering (for more details, see [1–5]).

On the other hand, uncertainty is inherent in most dynamical systems in its input, output, and manner, and fuzziness is a kind of uncertainty very common in real-world problems [6]. In 1965, Zadeh introduced the concepts of fuzzy sets and fuzzy numbers in [7], followed up in 1972 by Chang and Zadeh when they proposed the concept of the fuzzy derivative in [8]. A large number of researches have been studied in various aspects of the theory and applications of these notions; one of these research lines has been the fuzzy optimal control problem. In the past few decades, the fuzzy optimal control problem has received growing attention, and many results of researches have been reported in the literature ([9–19] and references therein).

Recently, a lot of works done in the field of the fuzzy optimal control problem have only examined problems with one control and one dependent state variable; however, many times, we will wish to examine fuzzy optimal control problems which arise in a wide variety of scientific and engineering applications such as physics, chemical engineering, and economy, with more variables (more controls and more states). It seems that it is a good idea to consider fuzzy optimal control problems of several variables and discuss how to handle such problems. Further, treating a special type of fuzzy optimal control problems such as problems having a type of constraint known as an isoperimetric constraint and problems involving higher order differential equations has been presented. In [11], the modified fuzzy Euler-Lagrange condition was established for the fuzzy Isoperimetric Variational Problem (IVP), which is considered as a fuzzy constrained variational problem, but, in this paper, we overcome the fuzzy optimal control problem involving the isoperimetric constraint, which is considered as a fuzzy constrained optimal control problem.

The main aim of this paper is to derive the necessary conditions of the fuzzy optimal control problem of several variables based on the concepts of differentiability and integrability of a fuzzy valued function parameterized by the left- and right-hand functions of its α-level set and variational approaches, in order to provide the solutions of this problem. However, the solutions of the fuzzy optimal control problem of several variables, optimal controls, and corresponding optimal states are not always fuzzy functions. Thus, to guarantee that the solutions of the fuzzy optimal control problem of several variables are always fuzzy functions, we will introduce the concepts of strong (fuzzy) and weak solutions of this problem.

The rest of this paper is organized as follows: In Section 2, we recall some basic terminologies and definitions used in the present paper. In Section 3, we establish our main results concerning the necessary conditions of the fuzzy optimal control problem of several variables and treating two special cases of the fuzzy optimal control problem. Additionally, we propose the definitions of strong (fuzzy) and weak solutions of our problem. In Section 4, we give three examples that can serve to illustrate our main results. In Section 5, we present some concluding remarks.

2. Preliminaries

Throughout this paper, $F(R)$ denotes the class of fuzzy subsets of the real axis. A fuzzy set $\tilde{v}$ on R is a mapping $\tilde{v}: R \longrightarrow [0,1]$. For each fuzzy set $\tilde{v}$, we denote its α-level set by $\tilde{v}[\alpha]$ and defined by $\tilde{v}[\alpha]=[v^l(\alpha), v^r(\alpha)]=\{x\in R : \tilde{v}(x)\geq \alpha\}$ for any $\alpha\in(0,1]$. The support of $\tilde{v}$ we denote by $\operatorname{supp}(\tilde{v})$, where $\operatorname{supp}\tilde{v}=\{x\in R:\tilde{v}(x)>0\}$. The closure of supp$\tilde{v}$ defines the 0-level set of $\tilde{v}$; thus,

$$\tilde{v}[\alpha]=\begin{cases}\{x\in R:\tilde{v}(x)\geq\alpha\}, & \text{if } 0<\alpha\leq 1,\\ \operatorname{cl}(\operatorname{supp}\tilde{v}), & \text{if } \alpha=0,\end{cases} \tag{1}$$

where $\operatorname{cl}(M)$ denotes the closure of set M. Fuzzy set $\tilde{v}\in F(R)$ is called a fuzzy number if

(1) $\tilde{v}$ is a normal fuzzy set, i.e., there exists an $x_0\in R$ such that $\tilde{v}(x_0)=1$

(2) $\tilde{v}$ is a convex fuzzy set, i.e., $\tilde{v}(rx+(1-r)y)\geq\min\{\tilde{v}(x),\tilde{v}(y)\}$ for any $x,y\in R$ and $r\in[0,1]$

(3) $\tilde{v}$ is upper semicontinuous on R

(4) $\tilde{v}[0]=\operatorname{cl}(\operatorname{supp}\tilde{v})=\operatorname{cl}(\bigcup_{\alpha\in(0,1]}\tilde{v}[\alpha])$ is compact

In the rest of this paper, we use E^1 to denote the fuzzy number space.

It is clear that the $\alpha-$level set $\tilde{v}[\alpha]=[v^l(\alpha),v^r(\alpha)]$ is bounded closed interval in R for all $\alpha\in[0,1]$, where $v^l(\alpha)$ and $v^r(\alpha)$ denote the left-hand and right-hand endpoints of $\tilde{v}[\alpha]$, respectively. Obviously, any $v\in R$ can be regarded as a fuzzy number $\tilde{v}$ defined by

$$\tilde{v}(x)=\begin{cases}1, & x=v,\\ 0, & x\neq v.\end{cases} \tag{2}$$

In particular, fuzzy zero is defined as $\tilde{0}(x)=1$ if $x=0$ and $\tilde{0}(x)=0$ otherwise.

Let $\tilde{a},\tilde{b}\in E^1$ and $k\in R$. For any $x\in R$, we can define the addition $\tilde{a}+\tilde{b}$ and scalar multiplication $k\tilde{a}$, respectively, as

$$\begin{aligned}\left(\tilde{a}+\tilde{b}\right)(x)&=\sup_{s+t=x}\min\left\{\tilde{a}(s),\tilde{b}(t)\right\},\\ (k\tilde{a})(x)&=\tilde{a}\left(\frac{x}{k}\right), k\neq 0,\\ (0\tilde{a})(x)&=\begin{cases}1, & x=0,\\ 0, & x\neq 0.\end{cases}\end{aligned} \tag{3}$$

Using $\alpha-$level set, we can also define the addition $\tilde{a}+\tilde{b}$ and scalar multiplication $k\tilde{a}$, respectively, as

$$\begin{aligned}\left(\tilde{a}+\tilde{b}\right)[\alpha]&=\tilde{a}[\alpha]+\tilde{b}[\alpha]=\left\{s+t: s\in\tilde{a}[\alpha], t\in\tilde{b}[\alpha]\right\},\\ (k\tilde{a})[\alpha]&=k\tilde{a}[\alpha]=\{kx: x\in\tilde{a}[\alpha]\}.\end{aligned} \tag{4}$$

Let $\tilde{a}$ be a fuzzy number, the opposite of $\tilde{a}$ is denoted by $-\tilde{a}$ and characterized by $-\tilde{a}(x)=\tilde{a}(-x)$ [20]. In the case that $\tilde{a}[\alpha]=[a^l(\alpha),a^r(\alpha)]$, we have $-\tilde{a}[\alpha]=[-a^r(\alpha),-a^l(\alpha)]$ for all $\alpha\in[0,1]$.

The binary operation "." in R can be extended to the binary operation "$\odot$" of two fuzzy numbers by using the extension principle. Let $\tilde{a}$ and $\tilde{b}$ be fuzzy numbers, then

$$\left(\tilde{\alpha}\odot\tilde{b}\right)(z)=\sup_{s\cdot t=z}\min\left\{\tilde{\alpha}(s),\tilde{b}(t)\right\}. \tag{5}$$

Using α-level set, the product $(\tilde{\alpha}\odot\tilde{b})$ is defined by

$$\begin{aligned}\left(\tilde{\alpha}\odot\tilde{b}\right)[\alpha]=\Big[&\min\left\{\alpha^l(\alpha)b^l(\alpha),\alpha^l(\alpha)b^r(\alpha),\alpha^r(\alpha)b^l(\alpha),\alpha^r(\alpha)b^r(\alpha)\right\},\\ &\max\left\{\alpha^l(\alpha)b^l(\alpha),\alpha^l(\alpha)b^r(\alpha),\alpha^r(\alpha)b^l(\alpha),\alpha^r(\alpha)b^r(\alpha)\right\}\Big],\end{aligned} \tag{6}$$

in the case that $\tilde{a}[\alpha]=[a^l(\alpha),a^r(\alpha)]$ and $\tilde{b}[\alpha]=[b^l(\alpha),b^r(\alpha)]$.

Lemma 1 (see [21]). *If $a^l:[0,1]\longrightarrow R$ and $a^r:[0,1]\longrightarrow R$ satisfy the following conditions:*

(1) a^l is a bounded increasing function

(2) a^r is a bounded decreasing function

(3) $a^l(1)\leq a^r(1)$

(4) $\lim_{\alpha\to k^-}a^l(\alpha)=a^l(k)$ and $\lim_{\alpha\to k^-}a^r(\alpha)=a^r(k)$, for all $0<k\leq 1$

(5) $\lim_{\alpha\to 0^+}a^l(\alpha)=a^l(0)$ and $\lim_{\alpha\to 0^+}a^r(\alpha)=a^r(0)$

then $\tilde{a}:R\longrightarrow[0,1]$ defined by $\tilde{a}(x)=\sup\{\alpha\mid a^l(\alpha)\leq x\leq a^r(\alpha)\}$ is a fuzzy number with $\tilde{a}[\alpha]=[a^l(\alpha),a^r(\alpha)]$. Conversely, if $\tilde{a}:R\longrightarrow[0,1]$ is a fuzzy number with $\tilde{a}[\alpha]=[a^l(\alpha),a^r(\alpha)]$, then the functions $a^l(\alpha)$ and $a^r(\alpha)$ satisfy conditions (1)-(5).

Define $D : E^1 \times E^1 \longrightarrow R_+ \cup \{0\}$ by

$$D\left(\tilde{a}, \tilde{b}\right) = \sup_{\alpha \in [0,1]} \max \left\{ |a^l(\alpha) - b^l(\alpha)|, |a^r(\alpha) - b^r(\alpha)| \right\}, \tag{7}$$

where $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$ and $\tilde{b}[\alpha] = [b^l(\alpha), b^r(\alpha)]$. $D(\tilde{a}, \tilde{b})$ is called the distance between fuzzy numbers $\tilde{a}$ and $\tilde{b}$.

It should be noted that D satisfies the following properties:

(1) (E^1, D) is a complete metric space

(2) $D(\tilde{a} + \tilde{c}, \tilde{b} + \tilde{c}) = D(\tilde{a}, \tilde{b})$

(3) $D(k\tilde{a}, k\tilde{b}) = |k| D(\tilde{a}, \tilde{b})$, where $\tilde{a}, \tilde{b}, \tilde{c} \in E^1$ and $k \in R$

A special class of fuzzy numbers is the class of triangular fuzzy numbers. We say that the fuzzy number $\tilde{a}$ is triangular if $a^l(1) = a^r(1)$, $a^l(\alpha) = a^l(1) - (1 - \alpha)(a^l(1) - a^l(0))$, and $a^r(\alpha) = a^l(1) + (1 - \alpha)(a^r(0) - a^l(1))$. The triangular fuzzy number $\tilde{a}$ is generally denoted by $\tilde{a} = (a^l(0), a^l(1), a^r(0))$.

Definition 2 (partial ordering [9]). *Let* $\tilde{a}, \tilde{b} \in E^1$, we write $\tilde{a} \preceq \tilde{b}$, *if* $a^l(\alpha) \le b^l(\alpha)$ and $a^r(\alpha) \le b^r(\alpha)$ *for all* $\alpha \in [0, 1]$. We also write $\tilde{a} \prec \tilde{b}$, *if* $\tilde{a} \preceq \tilde{b}$ and there exists $\alpha_0 \in [0, 1]$ such that $a^l(\alpha_0) < b^l(\alpha_0)$ *or* $a^r(\alpha_0) < b^r(\alpha_0)$. Furthermore, $\tilde{a} = \tilde{b}$, if $\tilde{a} \preceq \tilde{b}$ and $\tilde{a} \succeq \tilde{b}$. In other words, $\tilde{a} = \tilde{b}$, if $\tilde{a}[\alpha] = \tilde{b}[\alpha]$ for all $\alpha \in [0, 1]$.

In the sequel, we say that $\tilde{a}, \tilde{b} \in E^1$ are *comparable* if either $\tilde{a} \preceq \tilde{b}$ or $\tilde{a} \succeq \tilde{b}$ and *noncomparable* otherwise.

Definition 3 (gH-difference [22]). Suppose that $\tilde{a}, \tilde{b} \in E^1$, where $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$ and $\tilde{b}[\alpha] = [b^l(\alpha), b^r(\alpha)]$ for all $\alpha \in [0, 1]$, the generalized Hukuhara difference of two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ (gH-difference for short) is defined by

$$\tilde{\alpha} \ominus_{gH} \tilde{b} = \tilde{c} \Longleftrightarrow \begin{cases} (1)\tilde{\alpha} = \tilde{b} + \tilde{c}, \\ \text{or } (2)\tilde{b} = \tilde{\alpha} + (-1)\tilde{c}. \end{cases} \tag{8}$$

If $\tilde{c} = \tilde{\alpha} \ominus_{gH} \tilde{b}$ exists as a fuzzy number, then its α-level set is

$$\begin{aligned} c^l(\alpha) &= \min \left\{ a^l(\alpha) - b^l(\alpha), a^r(\alpha) - b^r(\alpha) \right\}, \\ c^r(\alpha) &= \max \left\{ a^l(\alpha) - b^l(\alpha), a^r(\alpha) - b^r(\alpha) \right\}, \end{aligned} \tag{9}$$

for all $\alpha \in [0, 1]$.

Definition 4 (fuzzy valued function [9]). The function $\tilde{f} : [t_0, t_1] \longrightarrow E^1$ is called a fuzzy valued function if $\tilde{f}(t)$ is assigned a fuzzy number for any $t \in [t_0, t_1]$. We also denote $\tilde{f}(t)[\alpha] = [f^l(t, \alpha), f^r(t, \alpha)]$, where $f^l(t, \alpha) = (\tilde{f}(t))^l(\alpha) = \min \{\tilde{f}(t)[\alpha]\}$ and $f^r(t, \alpha) = (\tilde{f}(t))^r(\alpha) = \max \{\tilde{f}(t)[\alpha]\}$. Therefore, any fuzzy valued function $\tilde{f}$ may be understood by $f^l(t, \alpha)$ and $f^r(t, \alpha)$ being, respectively, a bounded increasing function of α and a bounded decreasing function of α for $\alpha \in [0, 1]$. Also, it holds $f^l(t, \alpha) \le f^r(t, \alpha)$ for any $\alpha \in [0, 1]$.

Definition 5 (continuity of a fuzzy valued function [23]). We say that $\tilde{f} : [t_0, t_1] \longrightarrow E^1$ is continuous at $t \in [t_0, t_1]$, if both $f^l(t, \alpha)$ and $f^r(t, \alpha)$ are continuous functions at $t \in [t_0, t_1]$ for all $\alpha \in [0, 1]$.

Definition 6 (gH-differentiability of a fuzzy valued function [24]). Let $\hat{t} \in (t_0, t_1)$ and h be such that $\hat{t} + h \in (t_0, t_1)$, then the gH-derivative of a fuzzy valued function $\tilde{x}(t)$: $(t_0, t_1) \longrightarrow E^1$ at $\hat{t} \in (t_0, t_1)$ is defined as

$$\dot{\tilde{x}}\left(\hat{t}\right) = \lim_{h \to 0} \frac{\tilde{x}\left(\hat{t} + h\right) \ominus_{gH} \tilde{x}\left(\hat{t}\right)}{h}. \tag{10}$$

If $\dot{\tilde{x}}(\hat{t}) \in E^1$, we say that $\tilde{x}$ is generalized Hukuhara differentiable (gH-differentiable for short) at $\hat{t}$. Also, we say that $\tilde{x}$ is (1)-gH-differentiable at $\hat{t}$ if

$$(1) \quad \dot{\tilde{x}}\left(\hat{t}\right)[\alpha] = \left[\dot{x}^l\left(\hat{t}, \alpha\right), \dot{x}^r\left(\hat{t}, \alpha\right)\right], \text{for } \alpha \in [0, 1], \tag{11}$$

and $\tilde{x}$ is (2)-gH-differentiable at $\hat{t}$ if

$$(2) \quad \dot{\tilde{x}}\left(\hat{t}\right)[\alpha] = \left[\dot{x}^r\left(\hat{t}, \alpha\right), \dot{x}^l\left(\hat{t}, \alpha\right)\right], \text{for } \alpha \in [0, 1]. \tag{12}$$

Definition 7 (nth order gH-differentiability of a fuzzy valued function [25]). Let $\tilde{x}(t)$: $(t_0, t_1) \longrightarrow E^1$. We say that $\tilde{x}(t)$ is nth order gH-differentiable at $\hat{t}$ whenever the function $\tilde{x}(t)$ is gH-differentiable of the order i, $i = 1, 2, \cdots, n - 1$, at $\hat{t}$ and if there exist $\tilde{x}^{(n)}(\hat{t}) \in E^1$ such that

$$\tilde{x}^{(n)}\left(\hat{t}\right) = \lim_{h \to 0} \frac{\tilde{x}^{(n-1)}\left(\hat{t} + h\right) \ominus_{gH} \tilde{x}^{(n-1)}\left(\hat{t}\right)}{h}. \tag{13}$$

Definition 8 (switching point [26]). We say that a point $\hat{t}_0 \in (t_0, t_1)$ is a switching point for the differentiability of $\tilde{x}(t)$ if in any neighborhood N of $\hat{t}_0$ there exist points $\hat{t}_1 < \hat{t}_0 < \hat{t}_2$ such that

(i) type(I): at $\hat{t}_1$ (11) holds while (12) does not hold and at $\hat{t}_2$ (12) holds while (12) does not hold or

(ii) type(II): at $\hat{t}_1$ (12) holds while (12) does not hold and at $\hat{t}_2$ (11) holds while (12) does not hold

Definition 9 (see [21]). Let $\tilde{f} : [t_0, t_1] \longrightarrow E^1$. We say that $\tilde{f}$ is fuzzy-Riemann integrable to $I \in E^1$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[i, j]; \xi\}$ of $[t_0, t_1]$ with the norms $\Delta(p) < \delta$, we have

$$D\left(\sum_P^{*} (i - j) \odot \tilde{f}(\xi); I\right) < \epsilon, \tag{14}$$

where Σ^* denotes the fuzzy summation. We choose to write $I := \int_{t_0}^{t_1} \tilde{f}(t)dt$. Furthermore, for any $\alpha \in [0, 1]$,

$$\int_{t_0}^{t_1} \tilde{f}(t)[\alpha]dt = \left[\int_{t_0}^{t_1} f^l(t,\alpha)dt, \int_{t_0}^{t_1} f^r(t,\alpha)dt\right]. \quad (15)$$

Theorem 10 (see [24]). *If $\tilde{f}(t)$: $[t_0, t_1] \longrightarrow E^1$ is gH-differentiable with no switching point in the interval $[t_0, t_1]$, then we have*

$$\tilde{F}(t) = \int_{t_0}^{t_1} \dot{\tilde{f}}(x)dx = \tilde{f}(t_1) \ominus_{gH} \tilde{f}(t_0). \quad (16)$$

Theorem 11 (see [24]). *Let $\tilde{f}(t)$: $[t_0, t_1] \longrightarrow E^1$ be a continuous fuzzy valued function. Then,*

$$\tilde{F}(t) = \int_{t_0}^{t} \tilde{f}(x)dx, \quad t \in [t_0, t_1], \quad (17)$$

is gH-differentiable and $\dot{\tilde{F}}(t) = \tilde{f}(t)$.

From now, we use $C_{E^1}[t_0, t_1]$ to denote the space of all fuzzy valued functions that have continuous gH-derivatives on $[t_0, t_1]$ and $C_{E^1}^n[t_0, t_1]$ to denote the space of all fuzzy valued functions that have nth continuous gH-derivatives on $[t_0, t_1]$.

3. Fuzzy Optimal Control of Several Variables

This section is aimed at deriving the necessary conditions for the fuzzy optimal control problem of several variables. For this purpose, the fuzzy optimal control problem of several variables is introduced at first, then using fuzzy variational approaches, the problem is solved.

Consider the following fuzzy optimal control problem of several variables:

$$\begin{aligned}
\min_{\tilde{u}_1,\cdots,\tilde{u}_m} \quad & \tilde{J}(\tilde{x}_1, \cdots, \tilde{x}_n, \tilde{u}_1, \cdots, \tilde{u}_m) \\
& = \tilde{\phi}(\tilde{x}_1(t_1), \cdots, \tilde{x}_n(t_1), t_1) \\
& \quad + \int_{t_0}^{t_1} \tilde{f}(\tilde{x}_1(t), \cdots, \tilde{x}_n(t), \tilde{u}_1(t), \cdots, \tilde{u}_m(t), t)dt \\
\text{subject to} \quad & \dot{\tilde{x}}_j(t) = \tilde{g}_j(\tilde{x}_1(t), \cdots, \tilde{x}_n(t), \tilde{u}_1(t), \cdots, \tilde{u}_m(t), t) \\
& \tilde{x}_j(t_0) = \tilde{x}_{j0}, \quad \tilde{x}_j(t_1) \text{ is free, for } j = 1, 2, \cdots, n,
\end{aligned} \quad (18)$$

where $\tilde{f}, \tilde{g} : E^n \times E^m \times R \longrightarrow E^1$ are assumed to be functions of class $C_{E^1}[t_0, t_1]$ with respect to all their arguments. The fuzzy states $\tilde{x}_j(t)$ for $j = 1, 2, \cdots, n$ and the fuzzy controls $\tilde{u}_k(t)$ for $k = 1, 2, \cdots, m$ are functions of $t \in [t_0, t_1] \subseteq R$. Furthermore, $\tilde{x}_j(t)$ are assumed to be (1)-gH-differentiable functions for all $j = 1, 2, \cdots, n$. In problem (18), we make no requirements on n and m. In other words, $n > m$, $n = m$, or $n < m$ are all acceptable.

We say that an admissible fuzzy curve $(\tilde{x}_1^*, \cdots, \tilde{x}_n^*, \tilde{u}_1^*, \cdots, \tilde{u}_m^*)$ is the solution of problem (18), if for all admissible curve $(\tilde{x}_1, \cdots, \tilde{x}_n, \tilde{u}_1, \cdots, \tilde{u}_m)$ of problem (18)

$$\tilde{J}(\tilde{x}_1^*, \cdots, \tilde{x}_n^*, \tilde{u}_1^*, \cdots, \tilde{u}_m^*) \preceq \tilde{J}(\tilde{x}_1, \cdots, \tilde{x}_n, \tilde{u}_1, \cdots, \tilde{u}_m). \quad (19)$$

It is well know that, from Definition 2, the above inequality holds if and only if

$$\begin{aligned}
& J^l\left(x_1^{*^l}, x_1^{*^r}, \cdots, x_n^{*^l}, x_n^{*^r}, u_1^{*^l}, u_1^{*^r}, \cdots, u_m^{*^l}, u_m^{*^r}, t, \alpha\right) \\
& \quad \le J^l\left(x_1^l, x_1^r, \cdots, x_n^l, x_n^r, u_1^l, u_1^r, \cdots, u_m^l, u_m^r, t, \alpha\right), \\
& J^r\left(x_1^{*^l}, x_1^{*^r}, \cdots, x_n^{*^l}, x_n^{*^r}, u_1^{*^l}, u_1^{*^r}, \cdots, u_m^{*^l}, u_m^{*^r}, t, \alpha\right) \\
& \quad \le J^r\left(x_1^l, x_1^r, \cdots, x_n^l, x_n^r, u_1^l, u_1^r, \cdots, u_m^l, u_m^r, t, \alpha\right),
\end{aligned} \quad (20)$$

for all $\alpha \in [0, 1]$, where the $\alpha -$ level set of fuzzy curves $\tilde{x}^*, \tilde{x}, \tilde{u}^*$, and $\tilde{u}$ are characterized, respectively, by

$$\begin{aligned}
\tilde{x}_j^*(t)[\alpha] &= \left[x_j^{*^l}(t,\alpha), x_j^{*^r}(t,\alpha)\right], \\
\tilde{x}_j(t)[\alpha] &= \left[x_j^l(t,\alpha), x_j^r(t,\alpha)\right], \\
\tilde{u}_k^*(t)[\alpha] &= \left[u_k^{*^l}(t,\alpha), u_k^{*^r}(t,\alpha)\right], \\
\tilde{u}_k(t)[\alpha] &= \left[u_k^l(t,\alpha), u_k^r(t,\alpha)\right],
\end{aligned} \quad (21)$$

for $j = 1, 2, \cdots, n$ and $k = 1, 2, \cdots, m$.

Definition 12 (fuzzy Hamiltonian function). We define fuzzy Hamiltonian function as

$$\begin{aligned}
& \mathbb{H}\left(x_1^l, x_1^r, \cdots, x_n^l, x_n^r, u_1^l, u_1^r, \cdots, u_m^l, u_m^r, \lambda_1^l, \cdots, \lambda_n^l, \lambda_1^r, \cdots, \lambda_n^r, t, \alpha\right) \\
& \quad = f^l\left(x_1^l, x_1^r, \cdots, x_n^l, x_n^r, u_1^l, u_1^r, \cdots, u_m^l, u_m^r, t, \alpha\right) \\
& \qquad + f^r\left(x_1^l, x_1^r, \cdots, x_n^l, x_n^r, u_1^l, u_1^r, \cdots, u_m^l, u_m^r, t, \alpha\right) \\
& \qquad + \sum_{i=1}^{n} \left(\lambda_i^l g_i^l\left(x_1^l, x_1^r, \cdots, x_n^l, x_n^r, u_1^l, u_1^r, \cdots, u_m^l, u_m^r, t, \alpha\right)\right. \\
& \qquad \left. + \lambda_i^r g_i^r\left(x_1^l, x_1^r, \cdots, x_n^l, x_n^r, u_1^l, u_1^r, \cdots, u_m^l, u_m^r, t, \alpha\right)\right).
\end{aligned} \quad (22)$$

Theorem 13 (necessary conditions for problem (18)). *Assume that $(\tilde{x}_1^*(t), \cdots, \tilde{x}_n^*(t))$ is a vector of admissible fuzzy states and $(\tilde{u}_1^*(t), \cdots, \tilde{u}_m^*(t))$ is a vector of admissible fuzzy controls. Then, the necessary conditions for $(\tilde{x}_1^*(t), \cdots, \tilde{x}_n^*(t))$ and $(\tilde{u}_1^*(t), \cdots, \tilde{u}_m^*(t))$ to be optimal solutions for (18) are*

$$\begin{aligned}
\dot{\lambda}_j^{*^l}(t,\alpha) = -\frac{\partial \mathbb{H}}{\partial x_j^l}\Big(& x_1^l, x_1^r, \cdots, x_n^l, x_n^r, u_1^l, u_1^r, \cdots, u_m^l, u_m^r, \\
& \cdot \lambda_1^l, \cdots, \lambda_n^l, \lambda_1^r, \cdots, \lambda_n^r, t, \alpha\Big),
\end{aligned} \quad (23)$$

$$\dot{\lambda}_j^{*^r}(t,\alpha) = -\frac{\partial \mathbb{H}}{\partial x_j^r}\Big(x_1^l, x_1^r, \cdots, x_n^l, x_n^r, u_1^l, u_1^r, \cdots, u_m^l, u_m^r, \lambda_1^l, \cdots, \cdot \lambda_n^l, \lambda_1^r, \cdots, \lambda_n^r, t, \alpha\Big), \quad (24)$$

$$0 = \frac{\partial \mathbb{H}}{\partial u_k^l}\Big(x_1^l, x_1^r, \cdots, x_n^l, x_n^r, u_1^l, u_1^r, \cdots, u_m^l, u_m^r, \cdot \lambda_1^l, \cdots, \lambda_n^l, \lambda_1^r, \cdots, \lambda_n^r, t, \alpha\Big), \quad (25)$$

$$0 = \frac{\partial \mathbb{H}}{\partial u_k^r}\Big(x_1^l, x_1^r, \cdots, x_n^l, x_n^r, u_1^l, u_1^r, \cdots, u_m^l, u_m^r, \cdot \lambda_1^l, \cdots, \lambda_n^l, \lambda_1^r, \cdots, \lambda_n^r, t, \alpha\Big), \quad (26)$$

$$\lambda_j^l(t_1,\alpha) = \left(\frac{\partial \phi^l}{\partial x_j^l} + \frac{\partial \phi^r}{\partial x_j^l}\right)\Bigg|_{t=t_1}, \quad (27)$$

$$\lambda_j^r(t_1,\alpha) = \left(\frac{\partial \phi^l}{\partial x_j^r} + \frac{\partial \phi^r}{\partial x_j^r}\right)\Bigg|_{t=t_1}, \quad (28)$$

for all $\alpha \in [0,1]$, $t \in [t_0, t_1]$, $j = 1, 2, \cdots, n$, *and* $k = 1, 2, \cdots, m$.

Proof. Let us first consider the variation of u_k^l, u_k^r, x_j^l, and x_j^r defined, respectively, by

$$\begin{aligned} u_k^l &= u_k^{*^l} + \delta u_k^l, \\ u_k^r &= u_k^{*^r} + \delta u_k^r, \\ x_j^l &= x_j^{*^l} + \delta x_j^l, \\ x_j^r &= x_j^{*^r} + \delta x_j^r, \end{aligned} \quad (29)$$

for $k = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. Using Theorems 10 and 11 and because the problem is a minimization problem, we can rewrite $\tilde{J}$ as

$$\tilde{J}_1(\tilde{x}_1, \cdots, \tilde{x}_n, \tilde{u}_1, \cdots, \tilde{u}_m) = \int_{t_0}^{t_1}\Big(\tilde{f}(\tilde{x}_1(t), \cdots, \tilde{x}_n(t), \tilde{u}_1(t), \cdots, \tilde{u}_m(t), t) + \frac{d}{dt}\Big(\tilde{\phi}(\tilde{x}_1(t), \cdots, \tilde{x}_n(t), t)\Big)\Big)dt. \quad (30)$$

The increment of $\tilde{J}_1$, denoted by $\Delta\tilde{J}_1$ is

$$\begin{aligned} \Delta\tilde{J}_1 &= \int_{t_0}^{t_1}\Big(\tilde{f}(\tilde{x}_1^* + \delta\tilde{x}_1, \cdots, \tilde{x}_n^* + \delta\tilde{x}_n, \tilde{u}_1^* + \delta\tilde{u}_1, \cdots, \tilde{u}_m^* + \delta\tilde{u}_m, t) + \frac{d}{dt}\Big(\tilde{\phi}(\tilde{x}_1^* + \delta\tilde{x}_1, \cdots, \tilde{x}_n^* + \delta\tilde{x}_n, t)\Big)\Big)dt \\ &\quad \cdot \ominus_{\mathrm{gH}} \int_{t_0}^{t_1}\Big(\tilde{f}(\tilde{x}_1^*(t), \cdots, \tilde{x}_n^*(t), \tilde{u}_1^*(t), \cdots, \tilde{u}_m^*(t), t) + \frac{d}{dt}\Big(\tilde{\phi}(\tilde{x}_1^*(t), \cdots, \tilde{x}_n^*(t), t)\Big)\Big)dt \\ &= \int_{t_0}^{t_1}\Big(\tilde{f}[\tilde{x}_1^* + \delta\tilde{x}_1, \cdots, \tilde{x}_n^* + \delta\tilde{x}_n, \tilde{u}_1^* + \delta\tilde{u}_1, \cdots, \tilde{u}_m^* + \delta\tilde{u}_m][\alpha] + \frac{d}{dt}\Big(\tilde{\phi}[\tilde{x}_1^* + \delta\tilde{x}_1, \cdots, \tilde{x}_n^* + \delta\tilde{x}_n][\alpha]\Big)\Big)dt \\ &\quad \cdot \ominus_{\mathrm{gH}} \int_{t_0}^{t_1}\Big(\tilde{f}[\tilde{x}_1^*, \cdots, \tilde{x}_n^*, \tilde{u}_1^*, \cdots, \tilde{u}_m^*][\alpha] + \frac{d}{dt}\Big(\tilde{\phi}[\tilde{x}_1^*, \cdots, \tilde{x}_n^*][\alpha]\Big)\Big)dt, \end{aligned} \quad (31)$$

where

$$\begin{aligned} [\tilde{x}_1^* + \delta\tilde{x}_1, \cdots, \tilde{x}_n^* + \delta\tilde{x}_n, \tilde{u}_1^* + \delta\tilde{u}_1, \cdots, \tilde{u}_m^* + \delta\tilde{u}_m][\alpha] &= \Big(x_1^{*^l} + \delta x_1^l, x_1^{*^r} + \delta x_1^r, \cdots, x_n^{*^l} + \delta x_n^l, x_n^{*^r} + \delta x_n^r, \\ &\qquad \cdot u_1^{*^l} + \delta u_1^l, u_1^{*^r} + \delta u_1^r, \cdots, u_m^{*^l} + \delta u_m^l, u_m^{*^r} + \delta u_m^r, t, \alpha\Big), \\ [\tilde{x}_1^* + \delta\tilde{x}_1, \cdots, \tilde{x}_n^* + \delta\tilde{x}_n][\alpha] &= \Big(x_1^{*^l} + \delta x_1^l, x_1^{*^r} + \delta x_1^r, \cdots, x_n^{*^l} + \delta x_n^l, x_n^{*^r} + \delta x_n^r, t, \alpha\Big), \\ [\tilde{x}_1^*, \cdots, \tilde{x}_n^*, \tilde{u}_1^*, \cdots, \tilde{u}_m^*][\alpha] &= \Big(x_1^{*^l}, x_1^{*^r}, \cdots, x_n^{*^l}, x_n^{*^r}, u_1^{*^l}, u_1^{*^r}, \cdots, u_m^{*^l}, u_m^{*^r}, t, \alpha\Big), \\ [\tilde{x}_1^*, \cdots, \tilde{x}_n^*][\alpha] &= \Big(x_1^{*^l}, x_1^{*^r}, \cdots, x_n^{*^l}, x_n^{*^r}, t, \alpha\Big), \\ \Delta\tilde{J}_1[\alpha] &= \Big[\Delta J_1^l, \Delta J_1^r\Big]. \end{aligned} \quad (32)$$

Using the gH-difference, and without sake of generality, we consider

$$\begin{aligned}\Delta J_1^l &= \int_{t_0}^{t_1}\Big(f^l\Big(x_1^{*^l}+\delta x_1^l, x_1^{*^r}+\delta x_1^r,\cdots,x_n^{*^l}+\delta x_n^l, x_n^{*^r}+\delta x_n^r, u_1^{*^l}+\delta u_1^l, u_1^{*^r}+\delta u_1^r,\cdots,u_m^{*^l}+\delta u_m^l, u_m^{*^r}+\delta u_m^r, t,\alpha\Big)\\
&\quad+\frac{d}{dt}\Big(\phi^l\Big(x_1^{*^l}+\delta x_1^l, x_1^{*^r}+\delta x_1^r,\cdots,x_n^{*^l}+\delta x_n^l, x_n^{*^r}+\delta x_n^r,t,\alpha\Big)\Big)\Big)dt\\
&\quad-\int_{t_0}^{t_1}\Big(f^l\Big(x_1^{*^l},x_1^{*^r},\cdots,x_n^{*^l},x_n^{*^r},u_1^{*^l},u_1^{*^r},\cdots,u_m^{*^l},u_m^{*^r},t,\alpha\Big)+\frac{d}{dt}\Big(\phi^l\Big(x_1^{*^l},x_1^{*^r},\cdots,x_n^{*^l},x_n^{*^r},t,\alpha\Big)\Big)\Big)dt,\\
\Delta J_1^r &= \int_{t_0}^{t_1}\Big(f^r\Big(x_1^{*^l}+\delta x_1^l, x_1^{*^r}+\delta x_1^r,\cdots,x_n^{*^l}+\delta x_n^l, x_n^{*^r}+\delta x_n^r, u_1^{*^l}+\delta u_1^l, u_1^{*^r}+\delta u_1^r,\cdots,u_m^{*^l}+\delta u_m^l, u_m^{*^r}+\delta u_m^r, t,\alpha\Big)\\
&\quad+\frac{d}{dt}\Big(\phi^l\Big(x_1^{*^l}+\delta x_1^l, x_1^{*^r}+\delta x_1^r,\cdots,x_n^{*^l}+\delta x_n^l, x_n^{*^r}+\delta x_n^r,t,\alpha\Big)\Big)\Big)dt\\
&\quad-\int_{t_0}^{t_1}\Big(f^r\Big(x_1^{*^l},x_1^{*^r},\cdots,x_n^{*^l},x_n^{*^r},u_1^{*^l},u_1^{*^r},\cdots,u_m^{*^l},u_m^{*^r},t,\alpha\Big)+\frac{d}{dt}\Big(\phi^r\Big(x_1^{*^l},x_1^{*^r},\cdots,x_n^{*^l},x_n^{*^r},t,\alpha\Big)\Big)\Big)dt.\end{aligned}\tag{33}$$

Since $\tilde{J}_1(\tilde{x}_1^*,\cdots,\tilde{x}_n^*,\tilde{u}_1^*,\cdots,\tilde{u}_m^*)\le\tilde{J}_1(\tilde{x}_1,\cdots,\tilde{x}_n,\tilde{u}_1,\cdots,\tilde{u}_m)$ if and only if

$$\begin{aligned}J_1^l[\tilde{x}_1^*,\cdots,\tilde{x}_n^*,\tilde{u}_1^*,\cdots,\tilde{u}_m^*][\alpha]&\le J_1^l[\tilde{x}_1,\cdots,\tilde{x}_n,\tilde{u}_1,\cdots,\tilde{u}_m][\alpha],\\
J_1^r[\tilde{x}_1^*,\cdots,\tilde{x}_n^*,\tilde{u}_1^*,\cdots,\tilde{u}_m^*][\alpha]&\le J_1^r[\tilde{x}_1,\cdots,\tilde{x}_n,\tilde{u}_1,\cdots,\tilde{u}_m][\alpha],\end{aligned}\tag{34}$$

for all $\alpha\in[0,1]$, then $[\tilde{x}_1^*,\cdots,\tilde{x}_n^*,\tilde{u}_1^*,\cdots,\tilde{u}_m^*][\alpha]$ are optimal solutions for the crisp functions J_1^l and J_1^r. Suppose that δJ_1^l and δJ_1^r denote the first variation of J_1^l and J_1^r, respectively. From the classical theory of optimal control, we know that if $u_1^{*^l}, u_1^{*^r},\cdots,u_m^{*^l},u_m^{*^r}$ are optimal, then it is necessary that δJ_1^l and δJ_1^r are zero. In order to find the first variation of J_1^l and J_1^r, we need to evaluate the derivatives in the integrand of J_1^l and J_1^r along the optimal trajectory; then, we obtain

$$\begin{aligned}\Delta J_1^l &= \int_{t_0}^{t_1}\left[\sum_{j=1}^{n}\left(\frac{\partial f^l}{\partial x_j^l}\delta x_j^l+\frac{\partial f^l}{\partial x_j^r}\delta x_j^r\right)+\sum_{k=1}^{m}\left(\frac{\partial f^l}{\partial u_k^l}\delta u_k^l+\frac{\partial f^l}{\partial u_k^r}\delta u_k^r\right)\right.\\
&\quad\left.+\frac{d}{dt}\sum_{j=1}^{n}\left(\frac{\partial\phi^l}{\partial x_j^l}\delta x_j^l+\frac{\partial\phi^l}{\partial x_j^r}\delta x_j^r\right)\right]dt+O\left(\left(\delta u_1^{*^l}\right)^2\right)\\
&\quad+O\left(\left(\delta u_1^{*^r}\right)^2\right)+\cdots+O\left(\left(\delta u_m^{*^l}\right)^2\right)+O\left(\left(\delta u_m^{*^r}\right)^2\right),\\
\Delta J_1^r &= \int_{t_0}^{t_1}\left[\sum_{j=1}^{n}\left(\frac{\partial f^r}{\partial x_j^l}\delta x_j^l+\frac{\partial f^r}{\partial x_j^r}\delta x_j^r\right)+\sum_{k=1}^{m}\left(\frac{\partial f^r}{\partial u_k^l}\delta u_k^l+\frac{\partial f^r}{\partial u_k^r}\delta u_k^r\right)\right.\\
&\quad\left.+\frac{d}{dt}\sum_{j=1}^{n}\left(\frac{\partial\phi^r}{\partial x_j^l}\delta x_j^l+\frac{\partial\phi^r}{\partial x_j^r}\delta x_j^r\right)\right]dt+O\left(\left(\delta u_1^{*^l}\right)^2\right)\\
&\quad+O\left(\left(\delta u_1^{*^r}\right)^2\right)+\cdots+O\left(\left(\delta u_m^{*^l}\right)^2\right)+O\left(\left(\delta u_m^{*^r}\right)^2\right).\end{aligned}\tag{35}$$

Subsequently, on optimal trajectories, the first variation of J_1^l and J_1^r is zero, i.e.,

$$\begin{aligned}\delta J_1^l &= \int_{t_0}^{t_1}\left[\sum_{j=1}^{n}\left(\frac{\partial f^l}{\partial x_j^l}\delta x_j^l+\frac{\partial f^l}{\partial x_j^r}\delta x_j^r\right)+\sum_{k=1}^{m}\left(\frac{\partial f^l}{\partial u_k^l}\delta u_k^l+\frac{\partial f^l}{\partial u_k^r}\delta u_k^r\right)\right.\\
&\quad\left.+\frac{d}{dt}\sum_{j=1}^{n}\left(\frac{\partial\phi^l}{\partial x_j^l}\delta x_j^l+\frac{\partial\phi^l}{\partial x_j^r}\delta x_j^r\right)\right]dt=0,\\
\delta J_1^r &= \int_{t_0}^{t_1}\left[\sum_{j=1}^{n}\left(\frac{\partial f^r}{\partial x_j^l}\delta x_j^l+\frac{\partial f^r}{\partial x_j^r}\delta x_j^r\right)+\sum_{k=1}^{m}\left(\frac{\partial f^r}{\partial u_k^l}\delta u_k^l+\frac{\partial f^r}{\partial u_k^r}\delta u_k^r\right)\right.\\
&\quad\left.+\frac{d}{dt}\sum_{j=1}^{n}\left(\frac{\partial\phi^r}{\partial x_j^l}\delta x_j^l+\frac{\partial\phi^r}{\partial x_j^r}\delta x_j^r\right)\right]dt=0,\end{aligned}\tag{36}$$

for all variations. Now, we are ready to introduce the fuzzy Lagrange multiplier functions $\tilde{\lambda}_1(t),\cdots,\tilde{\lambda}_n(t)$ by considering the integral

$$\tilde{\psi}=\int_{t_0}^{t_1}\tilde{\lambda}(t)\odot\left(\tilde{g}_j(\tilde{x}_1(t),\cdots,\tilde{x}_n(t),\tilde{u}_1(t),\cdots,\tilde{u}_m(t),t)\ominus_{\mathrm{gH}}\dot{\tilde{x}}_j(t)\right)dt,\tag{37}$$

for $j=1,2,\cdots,n$. Using the gH-difference and without sake of generality, we consider the $\alpha-$level set of $\tilde{\psi}$, respectively, as

$$\psi^l=\int_{t_0}^{t_1}-\lambda_j^l\left(\dot{x}_j^l-g_j^l\right)dt,\tag{38}$$

$$\psi^r=\int_{t_0}^{t_1}-\lambda_j^r\left(\dot{x}_j^r-g_j^r\right)dt,\tag{39}$$

for $j=1,2,\cdots,n$. In the remainder of the proof, we will ignore similar arguments. We start by computing the variation of (38):

$$\delta\psi^l = \int_{t_0}^{t_1} \left(-\delta\lambda_j^l\left(\dot{x}_j^l - g_j^l\right) - \delta\left(\dot{x}_j^l - g_j^l\right)\lambda_j^l\right)dt, \quad \text{for } j = 1, 2, \cdots, n \tag{40}$$

$$= \int_{t_0}^{t_1} \left[\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\lambda_j^l\left(\frac{\partial g_j^l}{\partial x_i^l}\delta x_i^l + \frac{\partial g_j^l}{\partial x_i^r}\delta x_i^r\right) + \sum_{k=1}^{m}\lambda_j^l\left(\frac{\partial g_j^l}{\partial u_k^l}\delta u_k^l + \frac{\partial g_j^l}{\partial u_k^r}\delta u_k^r\right)\right) - \sum_{j=1}^{n}\lambda_j^l\delta\dot{x}_j^l\right]dt. \tag{41}$$

It is clear that from the definition of gH-difference, the first term of Equation (40) is zero. Integrating the last term on the RHS of (41) by parts and because $\tilde{x}_j(t_0)$ is specified, i.e., $\delta x_j^l(t_0) = 0$ (and $\delta x_j^r(t_0) = 0$) for all $j = 1, 2, \cdots, n$, then, we arrive at

$$\delta\psi^l = \int_{t_0}^{t_1} \left[\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\lambda_j^l\left(\frac{\partial g_j^l}{\partial x_i^l}\delta x_i^l + \frac{\partial g_j^l}{\partial x_i^r}\delta x_i^r\right) + \sum_{k=1}^{m}\lambda_j^l\left(\frac{\partial g_j^l}{\partial u_k^l}\delta u_k^l + \frac{\partial g_j^l}{\partial u_k^r}\delta u_k^r\right)\right) + \sum_{j=1}^{n}\dot{\lambda}_j^l\delta x_j^l\right]dt - \sum_{j=1}^{n}\lambda_j^l(t_1)\delta x_j^l(t_1). \tag{42}$$

By considering the following summations,

$$\begin{aligned}
&\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\lambda_j^l\left(\frac{\partial g_j^l}{\partial x_i^l}\delta x_i^l + \frac{\partial g_j^l}{\partial x_i^r}\delta x_i^r\right)\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\lambda_i^l\left(\frac{\partial g_i^l}{\partial x_j^l}\delta x_j^l + \frac{\partial g_i^l}{\partial x_j^r}\delta x_j^r\right)\right),\\
&\sum_{j=1}^{n}\left(\sum_{k=1}^{m}\lambda_j^l\left(\frac{\partial g_j^l}{\partial u_k^l}\delta u_k^l + \frac{\partial g_j^l}{\partial u_k^r}\delta u_k^r\right)\right) = \sum_{k=1}^{m}\left(\sum_{i=1}^{n}\lambda_i^l\left(\frac{\partial g_i^l}{\partial u_k^l}\delta u_k^l + \frac{\partial g_i^l}{\partial u_k^r}\delta u_k^r\right)\right),
\end{aligned} \tag{43}$$

we can rewrite $\delta\psi^l$ as

$$\delta\psi^l = \int_{t_0}^{t_1} \left[\sum_{j=1}^{n}\sum_{i=1}^{n}\lambda_i^l\left(\frac{\partial g_i^l}{\partial x_j^l}\delta x_j^l + \frac{\partial g_i^l}{\partial x_j^r}\delta x_j^r\right) + \sum_{k=1}^{m}\sum_{i=1}^{n}\lambda_i^l\left(\frac{\partial g_i^l}{\partial u_k^l}\delta u_k^l + \frac{\partial g_i^l}{\partial u_k^r}\delta u_k^r\right) + \sum_{j=1}^{n}\dot{\lambda}_j^l\delta x_j^l\right]dt - \sum_{j=1}^{n}\lambda_j^l(t_1)\delta x_j^l(t_1). \tag{44}$$

Similarly, when we consider (39) with $\delta x_j^r(t_0) = 0$ for all $j = 1, 2, \cdots, n$, and the same summation with small change (the right-hand functions of $\tilde{g}, \tilde{\lambda}$ instead of the left-hand functions), we arrive at

$$\delta\psi^r = \int_{t_0}^{t_1} \left[\sum_{j=1}^{n}\sum_{i=1}^{n}\lambda_i^r\left(\frac{\partial g_i^r}{\partial x_j^l}\delta x_j^l + \frac{\partial g_i^r}{\partial x_j^r}\delta x_j^r\right) + \sum_{k=1}^{m}\sum_{i=1}^{n}\lambda_i^r\left(\frac{\partial g_i^r}{\partial u_k^l}\delta u_k^l + \frac{\partial g_i^r}{\partial u_k^r}\delta u_k^r\right) + \sum_{j=1}^{n}\dot{\lambda}_j^r\delta x_j^r\right]dt - \sum_{j=1}^{n}\lambda_j^r(t_1)\delta x_j^r(t_1). \tag{45}$$

Since $\psi^l = 0$ and $\psi^r = 0$ for all $u_1^l, u_1^r, \cdots, u_m^l, u_m^r$, then $\delta\psi^l = 0$ and $\delta\psi^r = 0$. Further, we can replace the conditions $\delta J_1^l = 0$ and $\delta J_1^r = 0$ by $\delta J_1^l + \delta\psi^l = 0$ and $\delta J_1^r + \delta\psi^r = 0$, respectively [10]. Then, we have

$$\begin{aligned}
\delta J_1^l + \delta\psi^l = \int_{t_0}^{t_1} \Bigg[&\sum_{j=1}^{n}\left[\left(\frac{\partial f^l}{\partial x_j^l} + \sum_{i=1}^{n}\lambda_i^l\frac{\partial g_i^l}{\partial x_j^l} + \dot{\lambda}_j^l\right)\delta x_j^l + \left(\frac{\partial f^l}{\partial x_j^r} + \sum_{i=1}^{n}\lambda_i^l\frac{\partial g_i^l}{\partial x_j^r}\right)\delta x_j^r\right]\\
&+ \sum_{k=1}^{m}\left[\left(\frac{\partial f^l}{\partial u_k^l} + \sum_{i=1}^{n}\lambda_i^l\frac{\partial g_i^l}{\partial u_k^l}\right)\delta u_k^l + \left(\frac{\partial f^l}{\partial u_k^r} + \sum_{i=1}^{n}\lambda_i^l\frac{\partial g_i^l}{\partial u_k^r}\right)\delta u_k^r\right]\\
&+ \frac{d}{dt}\sum_{j=1}^{n}\left(\frac{\partial\phi^l}{\partial x_j^l}\delta x_j^l + \frac{\partial\phi^l}{\partial x_j^r}\delta x_j^r\right)\Bigg]dt - \sum_{j=1}^{n}\lambda_j^l(t_1)\delta x_j^l(t_1),
\end{aligned} \tag{46}$$

$$\begin{aligned}
\delta J_1^r + \delta\psi^r = \int_{t_0}^{t_1} \Bigg[&\sum_{j=1}^{n}\left[\left(\frac{\partial f^r}{\partial x_j^l} + \sum_{i=1}^{n}\lambda_i^r\frac{\partial g_i^r}{\partial x_j^l}\right)\delta x_j^l + \left(\frac{\partial f^r}{\partial x_j^r} + \sum_{i=1}^{n}\lambda_i^r\frac{\partial g_i^r}{\partial x_j^r} + \dot{\lambda}_j^r\right)\delta x_j^r\right]\\
&+ \sum_{k=1}^{m}\left[\left(\frac{\partial f^r}{\partial u_k^l} + \sum_{i=1}^{n}\lambda_i^r\frac{\partial g_i^r}{\partial u_k^l}\right)\delta u_k^l + \left(\frac{\partial f^r}{\partial u_k^r} + \sum_{i=1}^{n}\lambda_i^r\frac{\partial g_i^r}{\partial u_k^r}\right)\delta u_k^r\right]\\
&+ \frac{d}{dt}\sum_{j=1}^{n}\left(\frac{\partial\phi^r}{\partial x_j^l}\delta x_j^l + \frac{\partial\phi^r}{\partial x_j^r}\delta x_j^r\right)\Bigg]dt - \sum_{j=1}^{n}\lambda_j^r(t_1)\delta x_j^r(t_1).
\end{aligned} \tag{47}$$

Summing Equations (46) and (47) and using the definition of the fuzzy Hamiltonian function, we have

$$\int_{t_0}^{t_1}\left[\frac{d}{dt}\sum_{j=1}^{n}\left(\frac{\partial\phi^l}{\partial x_j^l}\delta x_j^l+\frac{\partial\phi^l}{\partial x_j^r}\delta x_j^r+\frac{\partial\phi^r}{\partial x_j^l}\delta x_j^l+\frac{\partial\phi^r}{\partial x_j^r}\delta x_j^r\right)+\left(\frac{\partial\mathbb{H}}{\partial x_j^l}+\dot{\lambda}_j^l\right)\delta x_j^l+\left(\frac{\partial\mathbb{H}}{\partial x_j^r}+\dot{\lambda}_j^r\right)\delta x_j^r+\frac{\partial\mathbb{H}}{\partial u_k^l}\delta u_k^l+\frac{\partial\mathbb{H}}{\partial u_k^r}\delta u_k^r\right]dt-\sum_{j=1}^{n}\left(\lambda_j^l(t_1)\delta x_j^l(t_1)+\lambda_j^r(t_1)\delta x_j^r(t_1)\right)=0. \tag{48}$$

Integrating the first term of (48) considering $\delta x_j^l(t_0)=0$ and $\delta x_j^r(t_0)=0$ for all $j=1,2,\cdots,n$ and removing the terms involving δx_j^l, δx_j^r, δu_k^l, and δu_k^r, therefore, the necessary conditions follow.

Note 1. It should be noted that in problem (18), *if $\tilde{x}_j(t_1)$ is specified, then $\tilde{\lambda}_j(t)$ has no boundary conditions for all $j=1,2,\cdots,n$.*

3.1. Isoperimetric Constraints. In this part, we turn our attention to a special type of fuzzy optimal control problem, defined as

$$\begin{aligned}\min_{\tilde{u}}\quad & \tilde{J}(\tilde{u})=\tilde{\phi}(\tilde{x}(t_1),t_1)+\int_{t_0}^{t_1}\tilde{f}(\tilde{x}(t),\tilde{u}(t),t)dt\\ \text{subject to}\quad & \dot{\tilde{x}}(t)=\tilde{g}(\tilde{x}(t),\tilde{u}(t),t),\quad \tilde{x}(t_0)=\tilde{x}_0\\ & \int_{t_0}^{t_1}\tilde{h}(\tilde{x}(t),\tilde{u}(t),t)dt=\tilde{A}.\end{aligned} \tag{49}$$

This type of constraint is known as an isoperimetric constraint, where $\tilde{f}$, $\tilde{g}$, and $\tilde{h}$ are assumed to be functions of class $C_{E^1}[t_0,t_1]$ with respect to all their arguments. To establish the solution method for this type of problem, we convert this problem to a more familiar form, the fuzzy optimal control problem of several variables, by introducing a second state variable $\tilde{y}(t)$, and let

$$\tilde{y}(t)=\int_{t_0}^{t}\tilde{h}(\tilde{x}(s),\tilde{u}(s),s)ds. \tag{50}$$

Therefore, if we use Theorems 10 and 11, then we have

$$\begin{aligned}&\dot{\tilde{y}}(t)=\tilde{h}(\tilde{x}(t),\tilde{u}(t),t),\\ &\tilde{y}(t_0)=\tilde{0},\tilde{y}(t_1)=\tilde{A}.\end{aligned} \tag{51}$$

Subsequently, problem (49) is transformed into

$$\begin{aligned}\min_{\tilde{u}}\quad & \tilde{J}(\tilde{u})=\tilde{\phi}(\tilde{x}(t_1),t_1)+\int_{t_0}^{t_1}\tilde{f}(\tilde{x}(t),\tilde{u}(t),t)dt\\ \text{subject to}\quad & \dot{\tilde{x}}(t)=\tilde{g}(\tilde{x}(t),\tilde{u}(t),t),\quad \tilde{x}(t_0)=\tilde{x}_0\\ & \dot{\tilde{y}}(t)=\tilde{h}(\tilde{x}(t),\tilde{u}(t),t),\quad \tilde{y}(t_0)=\tilde{0},\ \tilde{y}(t_1)=\tilde{A}.\end{aligned} \tag{52}$$

This problem can now be solved using Theorem 13, i.e., the developed method for solving fuzzy optimal control problems of several variables.

3.2. Higher Order Differential Equations. Here, we deal with problems involving higher order differential equations. Consider the following problem:

$$\begin{aligned}\min_{\tilde{u}_1,\cdots,\tilde{u}_m}\quad & \tilde{J}(\tilde{u}_1,\cdots,\tilde{u}_m)\\ & =\int_{t_0}^{t_1}\tilde{f}\left(\tilde{x}(t),\tilde{x}^{(1)}(t),\cdots,\tilde{x}^{n}(t),\tilde{u}_1(t),\cdots,\tilde{u}_m(t),t\right)dt\\ \text{subject to}\quad & \tilde{x}^{(n+1)}(t)=\tilde{g}\left(\tilde{x}(t),\tilde{x}^{(1)}(t),\cdots,\tilde{x}^{n}(t),\tilde{u}_1(t),\cdots,\tilde{u}_m(t),t\right)\\ & \tilde{x}(t_0)=\tilde{\beta}_1,\tilde{x}^{(1)}(t_0)=\tilde{\beta}_2,\cdots,\tilde{x}^{n}(t_0)=\tilde{\beta}_{n+1},\end{aligned} \tag{53}$$

for $n>1$. Theorem 13 does not directly deal with this type of problem. But it is easy to convert this problem to the fuzzy optimal control problem of several variables by introducing $n+1$ state variables defined, respectively, by

$$\begin{aligned}&\tilde{x}_1(t)=\tilde{x}(t),\\ &\tilde{x}_2(t)=\tilde{x}^{(1)}(t),\cdots,\tilde{x}_{n+1}(t)=\tilde{x}^{(n)}(t).\end{aligned} \tag{54}$$

Then, problem (53) is transformed into

$$\begin{aligned}\min_{\tilde{u}_1,\cdots,\tilde{u}_m}\quad & \tilde{J}(\tilde{u}_1,\cdots,\tilde{u}_m)\\ & =\int_{t_0}^{t_1}\tilde{f}(\tilde{x}_1(t),\tilde{x}_2(t),\cdots,\tilde{x}_{n+1}(t),\tilde{u}_1(t),\cdots,\tilde{u}_m(t),t)dt\\ \text{subject to}\quad & \tilde{x}_1^{(1)}(t)=\tilde{x}_2(t),\tilde{x}_1(t_0)=\tilde{\beta}_1\\ & \tilde{x}_2^{(1)}(t)=\tilde{x}_3(t),\tilde{x}_2(t_0)=\tilde{\beta}_2\\ & \vdots\\ & \tilde{x}_n^{(1)}(t)=\tilde{x}_{n+1}(t),\tilde{x}_n(t_0)=\tilde{\beta}_n\\ & \tilde{x}_{n+1}^{(1)}(t)=\tilde{g}(\tilde{x}_1(t),\tilde{x}_2(t),\cdots,\tilde{x}_{n+1}(t),\tilde{u}_1(t),\cdots,\\ & \tilde{u}_m(t),t),\tilde{x}_{n+1}(t_0)=\tilde{\beta}_{n+1},\end{aligned} \tag{55}$$

which can be solved by using Theorem 13, i.e., the developed method for solving fuzzy optimal control problems of several variables.

3.3. The Strong (Fuzzy) and Weak Solutions. The developed method for solving the fuzzy optimal control problem of several variables provides the solutions of this problem, optimal fuzzy controls and corresponding fuzzy states, by solving the necessary conditions introduced in Theorem 13. Meanwhile, to guarantee that the solutions of the fuzzy optimal control problem of several variables are always fuzzy functions, we propose the concepts of strong (fuzzy) and weak solutions of this problem. In the following definition, based on the conditions (1) and (2) of Lemma 1, we introduce the definition of strong (fuzzy) and weak solutions of the fuzzy optimal control problem of several variables (18).

Definition 14 (strong (fuzzy) and weak solutions).

(1) *Strong (Fuzzy) Solution*. We say that $\tilde{u}_k^*(t)[\alpha]$ and $\tilde{x}_j^*(t)[\alpha]$ are strong(fuzzy) solutions of problem (18) if $u_k^{*^l}(t,\alpha), u_k^{*^r}(t,\alpha), x_j^{*^l}(t,\alpha)$, and $x_j^{*^r}(t,\alpha)$, for $k=1,2,\cdots,m$ and $j=1,2,\cdots,n$, obtained from (23), (24), (25), (26), (27), and (28) satisfy their related properties defined in the conditions (1) and (2) of Lemma 1, for all $t\in[t_0,t_1]$ and $\alpha\in[0,1]$.

(2) *Weak Solution*. We say that $\tilde{u}_k^*(t)[\alpha]$ and $\tilde{x}_j^*(t)[\alpha]$ are weak solutions of problem (18) if $u_k^{*^l}(t,\alpha), u_k^{*^r}(t,\alpha)$, $x_j^{*^l}(t,\alpha)$, and $x_j^{*^r}(t,\alpha)$, for $k=1,2,\cdots,m$ and $j=1,2,\cdots,n$, obtained from (23), (24), (25), (26), (27), and (28) do not satisfy their related properties in the conditions (1) and (2) of Lemma 1; thus, we define $\tilde{u}_k^*(t)[\alpha]$ and $\tilde{x}_j^*(t)[\alpha]$ as

$$\tilde{u}_k^*(t)[\alpha]=\begin{cases}\left[2u_k^{*^r}(t,1)-u_k^{*^l}(t,\alpha),u_k^{*^r}(t,\alpha)\right], & \text{if } u_k^{*^l},u_k^{*^r} \text{ are decreasing functions of } \alpha,\\ \left[u_k^{*^l}(t,\alpha),2u_k^{*^l}(t,1)-u_k^{*^r}(t,\alpha)\right], & \text{if } u_k^{*^l},u_k^{*^r} \text{ are increasing functions of } \alpha,\\ \left[u_k^{*^r}(t,\alpha),u_k^{*^l}(t,\alpha)\right], & \text{if } u_k^{*^l} \text{ is a decreasing function and } u_k^{*^r} \text{ is an increasing function of } \alpha,\end{cases}$$
$$\tilde{x}_j^*(t)[\alpha]=\begin{cases}\left[2x_j^{*^r}(t,1)-x_j^{*^l}(t,\alpha),x_j^{*^r}(t,\alpha)\right], & \text{if } x_j^{*^l},x_j^{*^r} \text{ are decreasing functions of } \alpha,\\ \left[x_j^{*^l}(t,\alpha),2x_j^{*^l}(t,1)-x_j^{*^r}(t,\alpha)\right], & \text{if } x_j^{*^l},x_j^{*^r} \text{ are increasing functions of } \alpha,\\ \left[x_j^{*^r}(t,\alpha),x_j^{*^l}(t,\alpha)\right], & \text{if } x_j^{*^l} \text{ is a decreasing function and } x_j^{*^r} \text{ is an increasing function of } \alpha,\end{cases} \quad (56)$$

for all $t\in[t_0,t_1]$, $\alpha\in[0,1]$, $k=1,2,\cdots,m$, and $j=1,2,\cdots,n$.

Note 2. In the next section, we will give three examples that can serve to illustrate our main results, more specifically, a discussion on the strong solutions (fuzzy solutions) of our problems.

4. Illustrative Examples

Example 1. Find the fuzzy control that

$$\begin{aligned}&\text{minimize} && \tilde{J}(\tilde{u}(t))=\int_0^1\left(\tilde{x}_1(t)+\tilde{x}_2(t)\ominus_{gH}\tilde{u}^2(t)\right)dt\\ &\text{subject to} && \dot{\tilde{x}}_1(t)=\tilde{x}_2,\tilde{x}_1(0)=\tilde{0},\tilde{x}_1(1)\text{ is free}\\ &&& \dot{\tilde{x}}_2(t)=(0,1,3)\tilde{u}(t),\tilde{x}_2(0)=\tilde{0},\tilde{x}_2(1)\text{ is free.}\end{aligned} \quad (57)$$

Solution 1. First, without loss of generality, we formulate the fuzzy Hamiltonian function as

$$\begin{aligned}H&=f^l+f^r+\lambda_1^lg_1^l+\lambda_2^lg_2^l+\lambda_1^rg_1^r+\lambda_2^rg_2^r\\ &=\left(x_1^l+x_2^l-u^{l^2}\right)+\left(x_1^r+x_2^r-u^{r^2}\right)+\lambda_1^lx_2^l+\lambda_2^l\left(\alpha u^l\right)\\ &\quad+\lambda_1^rx_2^r+\lambda_2^r((3-2\alpha)u^r).\end{aligned} \quad (58)$$

In fact, $\tilde{\phi}(\tilde{x}_1(1),\tilde{x}_2(1),1)=\tilde{0}$, then

$$\begin{aligned}&\left[\phi^l\left(x_1^l(1),x_1^r(1),x_2^l(1),x_2^r(1),1\right),\right.\\ &\left.\cdot\,\phi^r\left(x_1^l(1),x_1^r(1),x_2^l(1),x_2^r(1),1\right)\right]=[0,0].\end{aligned} \quad (59)$$

The necessary conditions for optimality of Theorem 13 with the initial conditions and the dynamical system of (57) give

$$\dot{x}_1^l=x_2^l,x_1^l(0)=0, \quad (60)$$

$$\dot{x}_1^r=x_2^r,x_1^r(0)=0, \quad (61)$$

$$\dot{x}_2^l=\alpha u^l,x_2^l(0)=0, \quad (62)$$

$$\dot{x}_2^r=(3-2\alpha)u^r,x_2^r(0)=0, \quad (63)$$

$$\dot{\lambda}_1^l = -\frac{\partial H}{\partial x_1^l} \Longrightarrow \dot{\lambda}_1^l = -1, \lambda_1^l(1) = 0, \quad (64)$$

$$\dot{\lambda}_1^r = -\frac{\partial H}{\partial x_1^r} \Longrightarrow \dot{\lambda}_1^r = -1, \lambda_1^r(1) = 0, \quad (65)$$

$$\dot{\lambda}_2^l = -\frac{\partial H}{\partial x_2^l} \Longrightarrow \dot{\lambda}_2^l = -\left(1 + \lambda_1^l\right), \lambda_2^l(1) = 0, \quad (66)$$

$$\dot{\lambda}_2^r = -\frac{\partial H}{\partial x_2^r} \Longrightarrow \dot{\lambda}_2^r = -(1 + \lambda_1^r), \lambda_2^r(1) = 0, \quad (67)$$

$$\frac{\partial H}{\partial u^l} = 0 \Longrightarrow -2u^l + \alpha\lambda_2^l = 0 \Longrightarrow u^l = \frac{\alpha}{2}\lambda_2^l, \quad (68)$$

$$\frac{\partial H}{\partial u^r} = 0 \Longrightarrow -2u^r + (3-2\alpha)\lambda_2^r = 0 \Longrightarrow u^r = \frac{(3-2\alpha)}{2}\lambda_2^r. \quad (69)$$

We solve Equations (60), (61), (62), (63), (64), (65), (66), (67), (68), and (69) analytically; then, we obtain

$$\begin{aligned}
\lambda_1^{*^l}(t,\alpha) &= 1 - t,\\
\lambda_1^{*^r}(t,\alpha) &= 1 - t,\\
\lambda_2^{*^l}(t,\alpha) &= \frac{t^2 - 4t + 3}{2},\\
\lambda_2^{*^r}(t,\alpha) &= \frac{t^2 - 4t + 3}{2},\\
u^{*^l}(t,\alpha) &= \frac{\alpha\left(t^2 - 4t + 3\right)}{4},\\
u^{*^r}(t,\alpha) &= \frac{(3-2\alpha)\left(t^2 - 4t + 3\right)}{4},\\
x_1^{*^l}(t,\alpha) &= \frac{\alpha^2\left(t^4 - 8t^3 + 18t^2\right)}{48},\\
x_1^{*^r}(t,\alpha) &= \frac{(3-2\alpha)^2\left(t^4 - 8t^3 + 18t^2\right)}{48},\\
x_2^{*^l}(t,\alpha) &= \frac{\alpha^2\left(t^3 - 6t^2 + 9t\right)}{12},\\
x_2^{*^r}(t,\alpha) &= \frac{(3-2\alpha)^2\left(t^3 - 6t^2 + 9t\right)}{12}.
\end{aligned} \quad (70)$$

We can easily show that $u^{*^l}(t,\alpha), x_1^{*^l}(t,\alpha)$, and $x_2^{*^l}(t,\alpha)$ are continuous increasing functions of α, and $u^{*^r}(t,\alpha), x_1^{*^r}(t,\alpha)$, and $x_2^{*^r}(t,\alpha)$ are continuous decreasing functions of α. Furthermore,

$$\begin{aligned}
u^{*^l}(t,1) = u^{*^r}(t,1) &= \frac{t^2 - 4t + 3}{2},\\
x_1^{*^l}(t,1) = x_1^{*^r}(t,1) &= \frac{t^4 - 8t^3 + 18t^2}{48},\\
x_2^{*^l}(t,1) = x_2^{*^r}(t,1) &= \frac{t^3 - 6t^2 + 9t}{12},
\end{aligned} \quad (71)$$

for all $t \in [0,1]$. Therefore, the α-level set of optimal fuzzy control $\tilde{u}^*(t)$ and optimal fuzzy states $\tilde{x}_1^*(t)$ and $\tilde{x}_2^*(t)$ is characterized, respectively, by

$$\begin{aligned}
\tilde{u}^*(t)[\alpha] &= \left[\frac{\alpha\left(t^2 - 4t + 3\right)}{4}, \frac{(3-2\alpha)\left(t^2 - 4t + 3\right)}{4}\right],\\
\tilde{x}_1^*(t)[\alpha] &= \left[\frac{\alpha^2\left(t^4 - 8t^3 + 18t^2\right)}{48}, \frac{(3-2\alpha)^2\left(t^4 - 8t^3 + 18t^2\right)}{48}\right],\\
\tilde{x}_2^*(t)[a] &= \left[\frac{\alpha^2\left(t^3 - 6t^2 + 9t\right)}{12}, \frac{(3-2\alpha)^2\left(t^3 - 6t^2 + 9t\right)}{12}\right],
\end{aligned} \quad (72)$$

for all $\alpha, t \in [0,1]$. Therefore, the above solutions are strong (fuzzy) solutions of problem (57).

Example 2. Find the fuzzy control that

$$\begin{aligned}
&\text{minimize} && \tilde{J}(\tilde{u}(t)) = \int_0^1 \left(\tilde{x}(t) \ominus_{\text{gH}} \frac{1}{2}\tilde{u}^2(t)\right) dt\\
&\text{subject to} && \dot{\tilde{x}}(t) = \tilde{u}(t), \tilde{x}(0) = (0,1,2), \tilde{x}_1(1) \text{ is free}\\
& && \int_0^1 \tilde{x}(t)dt = (2,4,6).
\end{aligned} \quad (73)$$

Solution 2. First, we introduce a second state variable $\tilde{y}(t)$ by

$$\tilde{y}(t) = \int_0^t \tilde{x}(s)ds; \quad (74)$$

then, problem (73) converts to

$$\begin{aligned}
& \tilde{J}(\tilde{u}(t)) = \int_0^1 \left(\tilde{x}(t) \ominus \frac{1}{2}\tilde{u}^2(t)\right) dt,\\
\text{subject to}\quad & \dot{\tilde{x}}(t) = \tilde{u}(t), \tilde{x}(0) = (0,1,2), \tilde{x}(1) \text{ is free}\\
& \dot{\tilde{y}}(t) = \tilde{x}(t), \tilde{y}(0) = \tilde{0}, \tilde{y}(1) = (2,4,6).
\end{aligned} \quad (75)$$

Second, without loss of generality, we formulate the fuzzy Hamiltonian function as

$$H = \left(x^l - \frac{1}{2}u^{l^2}\right) + \left(x^r - \frac{1}{2}u^{r^2}\right) + \lambda_1^l u^l + \lambda_2^l x^l + \lambda_1^r u^r + \lambda_2^r x^r. \quad (76)$$

It is clear that

$$\begin{aligned}
&\tilde{\phi}(\tilde{x}(1), 1)[\alpha] = [0, 0],\\
&(0,1,2)[\alpha] = [\alpha, 2-\alpha],\\
&(2,4,6)[\alpha] = [2+2\alpha, 6-2\alpha].
\end{aligned} \quad (77)$$

Now, applying Theorem 13, we find the following necessary conditions:

$$\dot{x}^l = u^l, x^l(0) = \alpha, \tag{78}$$

$$\dot{x}^r = u^r, x^r(0) = 2 - \alpha, \tag{79}$$

$$\dot{y}^l = x^l, y^l(0) = 0, \tag{80}$$

$$\dot{y}^r = x^r, y^r(0) = 0, \tag{81}$$

$$\dot{\lambda}_1^l = -\frac{\partial H}{\partial x^l} \Longrightarrow \dot{\lambda}_1^l = -\left(1 + \lambda_2^l\right), \lambda_1^l(1) = 0, \tag{82}$$

$$\dot{\lambda}_1^r = -\frac{\partial H}{\partial x^r} \Longrightarrow \dot{\lambda}_1^r = -(1 + \lambda_2^r), \lambda_1^r(1) = 0, \tag{83}$$

$$\dot{\lambda}_2^l = -\frac{\partial H}{\partial y^l} = 0 \Longrightarrow \lambda_2^l = K_1, \text{for some constant } K_1, \tag{84}$$

$$\dot{\lambda}_2^r = -\frac{\partial H}{\partial y^r} = 0 \Longrightarrow \lambda_2^r = K_2, \text{for some constant } K_2, \tag{85}$$

$$\frac{\partial H}{\partial u^l} = 0 \Longrightarrow -u^l + \lambda_1^l = 0 \Longrightarrow u^l = \lambda_2^l, \tag{86}$$

$$\frac{\partial H}{\partial u^r} = 0 \Longrightarrow -u^r + \lambda_1^r = 0 \Longrightarrow u^r = \lambda_2^r. \tag{87}$$

In the rest of the solution, we will ignore the similar cases; we only consider the left-hand functions of α – level set.

We begin by substituting (84) into differential Equation (82), and then, by solving the differential equation, we obtain

$$\lambda_1^l(t, \alpha) = (1 + K_1)(1 - t). \tag{88}$$

Substituting $\lambda_1^l(t, \alpha)$ into (86) gives

$$u^l(t, \alpha) = (1 + K_1)(1 - t). \tag{89}$$

After that, we substitute $u^l(t, \alpha)$ into Equation (78), then solve it with the appropriate condition, we obtain

$$x^l(t, \alpha) = (1 + K_1)\left(t - \frac{t^2}{2}\right) + \alpha. \tag{90}$$

By the same manner, we substitute $x^l(t, \alpha)$ into Equation (80) and solve it with the appropriate condition; then, we have

$$y^l(t, \alpha) = (1 + K_1)\left(\frac{t^2}{2} - \frac{t^3}{6}\right) + \alpha t. \tag{91}$$

To find the value of constant of integration K_1, we use the condition $y^l(1, \alpha) = 2 + 2\alpha$; thus,

$$(1 + K_1)\left(\frac{1}{2} - \frac{1}{6}\right) + \alpha = 2 + 2\alpha \Longrightarrow (1 + K_1) = 6 + 3\alpha. \tag{92}$$

Therefore, for all $\alpha, t \in [0, 1]$, we arrive at

$$\begin{aligned} u^{*^l}(t, \alpha) &= (6 + 3\alpha)(1 - t), \\ x^{*^l}(t, \alpha) &= (6 + 3\alpha)\left(t - \frac{t^2}{2}\right) + \alpha, \\ y^{*^l}(t, \alpha) &= (6 + 3\alpha)\left(\frac{t^2}{2} - \frac{t^3}{6}\right) + \alpha t. \end{aligned} \tag{93}$$

Similarly, if we consider the right-hand functions of α – level set and for all $\alpha, t \in [0, 1]$, we arrive at

$$\begin{aligned} u^{*^r}(t, \alpha) &= (12 - 3\alpha)(1 - t), \\ x^{*^r}(t, \alpha) &= (12 - 3\alpha)\left(t - \frac{t^2}{2}\right) + 2 - \alpha, \\ y^{*^r}(t, \alpha) &= (12 - 3\alpha)\left(\frac{t^2}{2} - \frac{t^3}{6}\right) + (2 - \alpha)t. \end{aligned} \tag{94}$$

We can easily show that $u^{*^l}(t, \alpha), x^{*^l}(t, \alpha)$, and $y^{*^l}(t, \alpha)$ are continuous increasing functions of α and $u^{*^r}(t, \alpha), x^{*^r}$ (t, α), and $y^{*^r}(t, \alpha)$ are continuous decreasing functions of α. Moreover,

$$\begin{aligned} u^{*^l}(t, 1) &= u^{*^r}(t, 1) = 9(1 - t), \\ x_1^{*^l}(t, 1) &= x_1^{*^r}(t, 1) = 9\left(t - \frac{t^2}{2}\right) + 1, \\ x_2^{*^l}(t, 1) &= x_2^{*^r}(t, 1) = 9\left(\frac{t^2}{2} - \frac{t^3}{6}\right) + t, \end{aligned} \tag{95}$$

for all $t \in [0, 1]$. Therefore, the α-level set of optimal fuzzy control $\tilde{u}^*(t)$ and optimal fuzzy states $\tilde{x}_1^*(t)$ and $\tilde{x}_2^*(t)$ is characterized, respectively, by

$$\begin{aligned} \tilde{u}^*(t)[\alpha] &= [(6 + 3\alpha)(1 - t), (12 - 3\alpha)(1 - t)], \\ \tilde{x}^*(t)[\alpha] &= \left[(6 + 3\alpha)\left(t - \frac{t^2}{2}\right) + \alpha, (12 - 3\alpha)\left(t - \frac{t^2}{2}\right) + 2 - \alpha\right], \\ \tilde{y}^*(t)[a] &= \left[(6 + 3\alpha)\left(\frac{t^2}{2} - \frac{t^3}{6}\right) + \alpha t, (12)\left(\frac{t^2}{2} - \frac{t^3}{6}\right) + (2 - \alpha)t\right], \end{aligned} \tag{96}$$

for all $\alpha, t \in [0, 1]$. Therefore, the above solutions are strong (fuzzy) solutions of problem (73).

Example 3. Find the fuzzy control that

$$\begin{aligned} &\text{minimize} \quad \tilde{J}(\tilde{u}(t)) = \int_0^1 \left(\tilde{x}(t) + \tilde{x}^{(1)}(t) \ominus_{\text{gH}} \tilde{u}^2(t)\right) dt \\ &\text{subject to} \quad \tilde{x}^{(2)}(t) = (0, 1, 3)\tilde{u}, \tilde{x}(0) = \tilde{0}, \tilde{x}^{(1)}(0) = \tilde{0}. \end{aligned} \tag{97}$$

Solution 3. We let $\tilde{x}_1(t) = \tilde{x}(t)$ and $\tilde{x}_2(t) = \tilde{x}^{(1)}(t)$, then problem (97) converted to problem (57) in Example 1, i.e., problem (97) has the same solutions of problem (57).

5. Conclusion

In summary, we proved the necessary conditions for optimality of the fuzzy optimal control problem of several variables. Also, fuzzy optimal control problems involving isoperimetric constraint and higher order differential equation have been considered. By introducing new variables, we transformed these problems into fuzzy optimal control problems of several variables in order to use the developed method to solve these problems. The definitions of strong (fuzzy) and weak solutions of our problems have been introduced. By three examples, we discussed and summarized the applicability of our main results of this paper.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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