Hindawi
Advances in Mathematical Physics
Volume 2017, Article ID 6128102, 7 pages
https://doi.org/10.1155/2017/6128102

Research Article

Rota-Baxter Operators on 3-Dimensional Lie Algebras and the Classical R-Matrices

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Received 6 September 2017; Accepted 27 November 2017; Published 18 December 2017

Academic Editor: Boris G. Konopelchenko

Our aim is to classify the Rota-Baxter operators of weight 0 on the 3-dimensional Lie algebra whose derived algebra's dimension is 2. We explicitly determine all Rota-Baxter operators (of weight zero) on the 3-dimensional Lie algebras g. Furthermore, we give the corresponding solutions of the classical Yang-Baxter equation in the 6-dimensional Lie algebras $g \ltimes_{\mathrm{ad}^*} g^*$ and the induced left-symmetry algebra structures on g.

1. Introduction

In physics, the Yang-Baxter equation is a consistency equation which was first introduced in the field of statistical mechanics. It depends on the idea that, in some scattering situations, particles may preserve their momentum while changing their quantum internal states. Rota-Baxter algebra started with the probability study and has since found applications in many areas of mathematics and physics, such as quasi-symmetric functions, number theory, dendriform algebras, and Yang-Baxter equations.

A Rota-Baxter operator (of weight zero) on an associative algebra A is defined to be a linear map $P : g \to g$ satisfying

$$P(x)P(y) = P(P(x)y + xP(y)), \quad \forall x, y \in A. \tag{1}$$

Rota-Baxter operators (on associative algebras) were introduced by Baxter to solve an analytic formula in probability [1–4]. It has been related to other areas in mathematics and mathematical physics [5–9]. A Rota-Baxter operator (of weight zero) on a Lie algebra $(g, [\cdot,\cdot])$ is a linear operator $P : g \to g$ such that

$$[P(x), P(y)] = P([P(x), y] + [x, P(y)]), \quad \forall x, y \in g. \tag{2}$$

In fact, a Rota-Baxter operator is also called the operator form of the classical Yang-Baxter equation [10–13]. Let g be a Lie algebra and $r = \sum_i a_i \otimes b_i \in g \otimes g$. r is called a classical R-matrix if it is a solution of the classical Yang-Baxter equation (CYBE) in g: that is,

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \tag{3}$$

in $U(g)$, where $U(g)$ is the universal enveloping algebra of g and

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1,$$
$$r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \tag{4}$$
$$r_{23} = \sum_i 1 \otimes a_i \otimes b_i.$$

Set $r^{21} = \sum_i b_i \otimes a_i$. It is easy to obtain that r is skew-symmetric if and only if $r = -r^{21}$. Semenov-Tian-Shansky proved in [14] that r is skew-symmetric and there is a nondegenerate symmetric invariant bilinear form on Lie algebra g; relation (2) is equivalent to relation (3) when the weight is zero. Furthermore, Rota-Baxter operators of weights 0 and 1 on a Lie algebra g give rise to solutions of CYBE on the double Lie algebra $g \ltimes_{\mathrm{ad}^*} g^*$ over the direct sum $g \oplus g^*$ of the Lie algebra g and its dual space g^* [12, 15, 16]. Moreover, we can get some solutions of CYBE in $g \ltimes_{\mathrm{ad}^*} g^*$ Lie algebras through Rota-Baxter operators of any weight on g.

In [12], the authors gave all Rota-Baxter operators (of weight zero) on 3-dimensional simple Lie algebra $\mathrm{sl}(2,\mathbb{C})$. The aim of this paper is to determine the Rota-Baxter operators (of weight zero) on the 3-dimensional Lie algebra which is not simple, and the dimension of its derived algebra is 2. We will determine the Rota-Baxter operators on the Lie algebra g and give a family of solutions of CYBE in $g \ltimes_{\mathrm{ad}^*} g^*$. This paper is organized as follows. In Section 2, we give the classification theorem of Rota-Baxter operators (of weight zero) on g. In Section 3, we give the corresponding solutions of CYBE in $g \ltimes_{\mathrm{ad}^*} g^*$. In Section 4, we give the corresponding left-symmetry structure on g.

2. The Rota-Baxter Operators on g (of Weight Zero)

2.1. Notations and the Classification Theorem. Let g be a 3-dimensional linear Lie algebra whose standard (Cartan-Weyl) basis consists of e_1, e_2, e_3 over the field of complex numbers $\mathbb{C}$ with the following Lie brackets:

$$\begin{aligned} [e_1, e_2] &= e_1, \\ [e_1, e_3] &= 0, \\ [e_2, e_3] &= e_1 + e_3. \end{aligned} \tag{5}$$

Thus, a linear operator $P : g \to g$ is determined by

$$\begin{pmatrix} P(e_1) \\ P(e_2) \\ P(e_3) \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{6}$$

where $b_{ij} \in \mathbb{C}$, $1 \le i, j \le 3$. P is a Rota-Baxter operator on g if the above matrix $(b_{ij})_{3\times 3}$ satisfies (2). Here is our main theorem.

Theorem 1. *All Rota-Baxter operators of weight zero on g are listed in their matrices form with respect to the Cartan-Weyl basis below, where a, b, and c are nonzero complex numbers.*

$$P_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & 1 & 0 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & -a^2 \\ 0 & 1 & -a \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 0 & 0 & 0 \\ ab & a & -a^2 \\ b & 1 & -a \end{pmatrix},$$

$$P_5 = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P_6 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P_9 = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & b \\ 1 & 0 & 0 \end{pmatrix},$$

$$P_{10} = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$P_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 1 & 0 & 0 \end{pmatrix},$$

$$P_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$P_{13} = \begin{pmatrix} 0 & 0 & 0 \\ a & 1 & b \\ 0 & 0 & 0 \end{pmatrix},$$

$$P_{14} = \begin{pmatrix} 0 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P_{15} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 0 \end{pmatrix},$$

$$P_{16} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P_{17} = \begin{pmatrix} 0 & 0 & 1 \\ a & 0 & b \\ 0 & 0 & -1 \end{pmatrix},$$

$$P_{18} = \begin{pmatrix} a & 0 & 1 \\ 0 & 0 & b \\ a^2 - a & 0 & a-1 \end{pmatrix} \quad \left(a \neq \frac{1}{2}\right),$$

$$P_{19} = \begin{pmatrix} a & 0 & 1 \\ b & 0 & 0 \\ a^2 - a & 0 & a-1 \end{pmatrix} \quad \left(a \neq \frac{1}{2}\right),$$

$$P_{20} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & a \\ 0 & 0 & -1 \end{pmatrix},$$

$$P_{21} = \begin{pmatrix} 0 & 0 & 1 \\ a & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$P_{22} = \begin{pmatrix} a & 0 & 1 \\ 0 & 0 & 0 \\ a^2 - a & 0 & a-1 \end{pmatrix} \quad \left(a \neq \frac{1}{2}\right),$$

$$P_{23} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$P_{24} = \begin{pmatrix} a & 0 & 1 \\ b & 0 & c \\ a^2 - a & 0 & a-1 \end{pmatrix} \quad \left(a \neq \frac{1}{2}\right),$$

$$P_{25} = \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ 0 & 0 & a \\ -\frac{1}{4} & 0 & -\frac{1}{2} \end{pmatrix},$$

$$P_{26} = \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ a & 0 & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{2} \end{pmatrix},$$

$$P_{27} = \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ 0 & 0 & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{2} \end{pmatrix},$$

$$P_{28} = \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ a & 0 & b \\ -\frac{1}{4} & 0 & -\frac{1}{2} \end{pmatrix},$$

$$P_{29} = \begin{pmatrix} 0 & 1 & 2a \\ 0 & a & 2a^2 \\ 0 & -\frac{1}{2} & -a \end{pmatrix},$$

$$P_{30} = \begin{pmatrix} a & 1 & 2a \\ 0 & 0 & 0 \\ -\frac{a}{2} & -\frac{1}{2} & -a \end{pmatrix},$$

$$P_{31} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \end{pmatrix},$$

$$P_{32} = \begin{pmatrix} a & 1 & 2(a+b) \\ ab & b & -2b^2 - 2ab \\ -\frac{a}{2} & -\frac{1}{2} & -(a+b) \end{pmatrix}. \tag{7}$$

2.2. Reduction to Quadratic Equations. In order to show that P is a Rota-Baxter operator, we only check the following:

$$\begin{aligned} [P(e_1), P(e_2)] &= P([P(e_1), e_2] + e_1, P(e_2)], \\ [P(e_2), P(e_3)] &= P([P(e_2), e_3] + e_2, P(e_3)], \\ [P(e_1), P(e_3)] &= P([P(e_1), e_3] + e_1, P(e_3)]. \end{aligned} \tag{8}$$

It follows from (5) and (6) that

$$\begin{aligned} [P(e_2), P(e_3)] = {} & (b_{21}b_{32} - b_{22}b_{31} + b_{22}b_{33} - b_{23}b_{32}) e_1 \\ & + (b_{22}b_{33} - b_{23}b_{32}) e_3, \end{aligned} \tag{9}$$

while

$$\begin{aligned} & P([P(e_2), e_3] + [e_2, P(e_3)]) \\ & \quad = P((b_{22} - b_{31} + b_{33}) e_1) - P((b_{22} + b_{33}) e_3) \\ & \quad = (b_{22}b_{11} - b_{31}b_{11} + b_{33}b_{11} + b_{22}b_{31} + b_{33}b_{31}) e_1 \\ & \qquad + (b_{22}b_{12} - b_{31}b_{12} + b_{33}b_{12} + b_{22}b_{32} + b_{33}b_{32}) e_2 \\ & \qquad + (b_{22}b_{13} - b_{31}b_{13} + b_{33}b_{13} + b_{22}b_{33} + b_{33}b_{33}) e_3. \end{aligned} \tag{10}$$

Comparing the coefficients in (9) and (10), we have

$$\begin{aligned} & b_{21}b_{32} - 2b_{22}b_{31} + b_{22}b_{33} - b_{23}b_{32} - b_{22}b_{11} + b_{31}b_{11} \\ & \quad - b_{33}b_{11} - b_{33}b_{31} = 0, \end{aligned} \tag{11}$$

$$b_{22}b_{12} - b_{31}b_{12} + b_{33}b_{12} + b_{22}b_{32} + b_{33}b_{32} = 0, \tag{12}$$

$$b_{23}b_{32} + b_{22}b_{13} - b_{31}b_{13} + b_{33}b_{13} + b_{33}b_{33} = 0. \tag{13}$$

Similarly, from

$$\begin{aligned} [P(e_1), P(e_2)] &= P([P(e_1), e_2] + e_1, P(e_2)], \\ [P(e_1), P(e_3)] &= P([P(e_1), e_3] + e_1, P(e_3)], \end{aligned} \tag{14}$$

we obtain the following six equations:

$$b_{11}b_{11} + b_{12}b_{21} - b_{13}b_{11} - b_{13}b_{31} - b_{12}b_{23} + b_{13}b_{22} = 0, \quad (15)$$

$$b_{11}b_{12} + b_{22}b_{12} - b_{13}b_{12} - b_{13}b_{32} = 0, \quad (16)$$

$$b_{11}b_{13} + 2b_{22}b_{13} - b_{13}b_{13} - b_{13}b_{33} - b_{12}b_{23} = 0, \quad (17)$$

$$b_{12}b_{33} - 2b_{12}b_{31} - b_{13}b_{32} - b_{12}b_{11} = 0, \quad (18)$$

$$b_{12}b_{12} + 2b_{32}b_{12} = 0, \quad (19)$$

$$b_{12}b_{13} + 2b_{32}b_{13} = 0. \quad (20)$$

2.3. Solving the Quadratic Equations. Equation (19) implies $b_{12}(b_{12} + 2b_{32}) = 0$. To solve the quadratic equations (11), (12), (13), (15), (16), (17), (18), (19), and (20), we distinguish the following cases depending on whether $b_{12} = 0$ or not.

Case 1. $b_{12} = 0, b_{12} + 2b_{32} \neq 0$. That is, $b_{12} = 0, b_{32} \neq 0$, taking $b_{32} = 1$. Equation (16) implies $b_{13} = 0$. Equation (15) implies $b_{11} = 0$. Equation (11) implies $b_{21} = b_{22}b_{31} + b_{22}^2 + b_{23}$. Equation (12) implies $b_{33} = -b_{22}$. Equation (13) implies $b_{23} = -b_{33}^2 = -b_{22}^2$. We obtain

$$P = \begin{pmatrix} 0 & 0 & 0 \\ b_{22}b_{31} & b_{22} & -b_{22}^2 \\ b_{31} & 1 & -b_{22} \end{pmatrix}. \quad (21)$$

Taking $b_{22} = 0, b_{31} = a$, we obtain P_1. Taking $b_{22} = a, b_{31} = 0$, we obtain P_2. Taking $b_{22} = 0, b_{31} = 0$, we obtain P_3. Taking $b_{22} = a, b_{31} = b$, we obtain P_4.

Case 2. Assume $b_{12} = 0, b_{12}+2b_{32} = 0$. That is, $b_{12} = 0, b_{32} = 0$. We distinguish the two cases depending on whether $b_{13} = 0$ or not.

Subcase 2.1. If $b_{13} = 0$, then (13) implies $b_{33} = 0$. Equation (15) implies $b_{11} = 0$. Equation (11) implies $b_{22}b_{31} = 0$.

Subcase 2.1.1. If $b_{22} = 0, b_{31} = 0$, we obtain

$$P = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & 0 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}. \quad (22)$$

Taking $b_{21} = a, b_{23} = 1$, we obtain P_5. Taking $b_{21} = 1, b_{23} = 0$, we obtain P_6. Taking $b_{21} = 0, b_{23} = 1$, we obtain P_7. Taking $b_{21} = 0, b_{23} = 0$, we obtain P_8.

Subcase 2.1.2. If $b_{22} = 0, b_{31} \neq 0$, taking $b_{31} = 1$, we obtain

$$P = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & 0 & b_{23} \\ 1 & 0 & 0 \end{pmatrix}. \quad (23)$$

Taking $b_{21} = a, b_{23} = b$, we obtain P_9. Taking $b_{21} = a, b_{23} = 0$, we obtain P_{10}. Taking $b_{21} = 0, b_{23} = a$, we obtain P_{11}. Taking $b_{21} = 0, b_{23} = 0$, we obtain P_{12}.

Subcase 2.1.3. If $b_{22} \neq 0, b_{31} = 0$, taking $b_{22} = 1$, we obtain

$$P = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & 1 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}. \quad (24)$$

Taking $b_{21} = a, b_{23} = b$, we obtain P_{13}. Taking $b_{21} = a, b_{23} = a$, we obtain P_{14}. Taking $b_{21} = 0, b_{23} = a$, we obtain P_{15}. Taking $b_{21} = 0, b_{23} = 0$, we obtain P_{16}.

Subcase 2.2. If $b_{13} \neq 0$, taking $b_{13} = 1$, (17) implies $b_{22} = (b_{33} - b_{11} + 1)/2$. Then (15) implies $b_{31} = (2b_{11}^2 - 3b_{11} + b_{33} + 1)/2$. Equation (13) implies $b_{11} + b_{33} = b_{11}^2 - b_{33}^2$. That is, $(b_{11} + b_{33})(b_{11} - b_{33} - 1) = 0$.

Subcase 2.2.1. If $b_{11} + b_{33} \neq 0, b_{11} - b_{33} - 1 = 0, (b_{11} \neq 1/2)$, and then $b_{22} = 0, b_{31} = b_{11}^2 - b_{11}$, we obtain

$$P = \begin{pmatrix} b_{11} & 0 & 1 \\ b_{21} & 0 & b_{23} \\ b_{11}^2 - b_{11} & 0 & b_{11} - 1 \end{pmatrix}, \quad \left(b_{11} \neq \frac{1}{2}\right). \quad (25)$$

Taking $b_{11} = 0$, $b_{21} = a, b_{23} = b$, we obtain P_{17}. Taking $b_{11} = a$, $b_{21} = 0, b_{23} = b$ $(a \neq 1/2)$, we obtain P_{18}. Taking $b_{11} = a$, $b_{21} = b, b_{23} = 0$ $(a \neq 1/2)$, we obtain P_{19}. Taking $b_{11} = 0$, $b_{21} = 0, b_{23} = a$, we obtain P_{20}. Taking $b_{11} = 0, b_{21} = a, b_{23} = 0$, we obtain P_{21}. Taking $b_{11} = a, b_{21} = 0, b_{23} = 0$ $(a \neq 1/2)$, we obtain P_{22}. Taking $b_{11} = 0, b_{21} = 0, b_{23} = 0$, we obtain P_{23}. Taking $b_{11} = a, b_{21} = b, b_{23} = c$ $(a \neq 1/2)$, we obtain P_{24}.

Subcase 2.2.2. If $b_{11}+b_{33} = 0, b_{11}-b_{33}-1 \neq 0$ $(b_{11} \neq 1/2)$, and then $b_{22} = (-2b_{11}+1)/2, b_{31} = (2b_{11}^2-4b_{11}+1)/2$, (3.4) implies $8b_{11}^3 - 12b_{11}^2 + 6b_{11} - 1 = 0$. Then we have $8(b_{11} - 1/2)^3 = 0$, $b_{11} = 1/2$, giving a contradiction.

Subcase 2.2.3. If $b_{11}+b_{33} = 0, b_{11}-b_{33}-1 = 0$, that is, $b_{11} = 1/2$, $b_{33} = -1/2$, and then $b_{22} = 0, b_{31} = -1/4$, we obtain

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ b_{21} & 0 & b_{23} \\ -\frac{1}{4} & 0 & -\frac{1}{2} \end{pmatrix}. \quad (26)$$

Taking $b_{21} = 0, b_{23} = a$, we obtain P_{25}. Taking $b_{21} = a$, $b_{23} = 0$, we obtain P_{26}. Taking $b_{21} = 0, b_{23} = 0$, we obtain P_{27}. Taking $b_{21} = a, b_{23} = b$, we obtain P_{28}.

Case 3. Assume $b_{12} \neq 0$, $(b_{12} + 2b_{32}) = 0$. Taking $b_{12} = 1$, then $b_{32} = -1/2$. Equation (16) implies $b_{13} = 2b_{11} + 2b_{22}$. Equations (12) and (18) imply $b_{31} = (b_{22} + b_{33})/2$. Equation (17) implies $b_{23} = -2b_{11}^2 - 2b_{11}b_{22} - 2b_{11}b_{33} - 2b_{22}b_{33}$. Equation (15) implies $b_{21} = -b_{11}b_{22} - b_{11}b_{33} - b_{22}b_{33} - b_{11}^2 - b_{22}^2$. Equations (11), (15), and (17) imply $b_{11}^2 + b_{22}^2 + b_{33}^2 + 2b_{11}b_{22} + 2b_{11}b_{33} + 2b_{22}b_{33} = 0$.

Then we have $(b_{11}+b_{22}+b_{33})^2 = 0$. So $b_{33} = -(b_{11}+b_{22})$, $b_{21} = b_{11}b_{22}$, $b_{23} = -2b_{22}^2 - 2b_{11}b_{22}$, $b_{31} = -b_{11}/2$. We obtain

$$P = \begin{pmatrix} b_{11} & 1 & 2b_{11}+2b_{22} \\ b_{11}b_{22} & b_{22} & -2b_{22}^2 - 2b_{11}b_{22} \\ -\frac{b_{11}}{2} & -\frac{1}{2} & -(b_{11}+b_{22}) \end{pmatrix}. \quad (27)$$

Taking $b_{11} = 0, b_{22} = a$, we obtain P_{29}. Taking $b_{11} = a, b_{22} = 0$, we obtain P_{30}. Taking $b_{11} = 0, b_{22} = 0$, we obtain P_{31}. Taking $b_{11} = a, b_{22} = b$, we obtain P_{32}.

3. Solutions of the CYBE in $g \ltimes_{ad^*} g^*$

In this section, we will give some solutions of CYBE in $g \ltimes_{\mathrm{ad}^*} g^*$. Let $(g, [,])$ be a Lie algebra and $\beta : g \to gl(V)$ a representation of g. On the vector space $g \oplus V$, there is natural Lie algebra structure (denoted by $g \ltimes_\beta V$) given by

$$[x_1 + v_1, x_2 + v_2] = [x_1, x_2] + \beta(x_1) v_2 - \beta(x_2) v_1, \quad x_1, x_2 \in g, \ v_1, v_2 \in V. \quad (28)$$

Let $\beta^* : g \to gl(V^*)$ be the dual representation of β. A linear map $P : V \to g$ can be identified as an element $\widetilde{P}$ in $g \otimes V^* \subset (g \ltimes_\beta^* V^*) \otimes (g \ltimes_\beta^* V^*)$ as follows. Let $\{v_1, v_2, \ldots, v_m\}$ be a basis of V and $\{v_1^*, v_2^*, \ldots, v_m^*\}$ the dual basis in V^*: that is, $v_i^*(v_j) = \delta_{ij}$. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of g. Set $P(v_i) = \sum_{i=1}^n a_{ij} e_j$, $1 \le i \le n$. Since, as a vector space, $\mathrm{Hom}(V, g) \cong g \otimes V^*$, then

$$\widetilde{P} = \sum_{i=1}^n P(v_i) \otimes v_i^* = \sum_{i=1}^m \sum_{j=1}^n a_{ij} e_j \otimes v_i^* \subseteq (g \ltimes_\beta^* V^*) \otimes (g \ltimes_\beta^* V^*). \quad (29)$$

Lemma 2 (see [15]). *Let g be a Lie algebra; let (V, β) be a g-module. A linear map $P : g \to g$ is a Rota-Baxter operator if and only if $r = P - P^{21}$ is a skew-symmetric solution of CYBE in $g \ltimes_{ad^*} g^*$.*

Now consider the adjoint representation of g, (g, ad) which is a g-module. Let e_1, e_2, e_3 be the Cartan-Weyl basis. Using Lemma 2 and relation (29), we can obtain a family of solutions of CYBE in $g \ltimes_{\mathrm{ad}^*} g^*$ through the Rota-Baxter operators on g given in Theorem 1.

Theorem 3. *The following tensors are solutions of the classical Yang-Baxter equation in $g \ltimes_{ad^*} g^*$, where a, b, and c are nonzero complex numbers*

$$r_1 = (ae_1 + e_2) \otimes e_3^* - e_3^* \otimes (ae_1 + e_2),$$

$$r_2 = (ae_2 - a^2 e_3) \otimes e_2^* + (e_2 - ae_3) \otimes e_3^* - e_2^* \otimes (ae_2 - a^2 e_3) - e_3^* \otimes (e_2 - ae_3),$$

$$r_3 = e_2 \otimes e_3^* - e_3^* \otimes e_2,$$

$$r_4 = (abe_1 + ae_2 - a^2 e_3) \otimes e_2^* + (be_1 + e_2 - ae_3) \otimes e_3^* - e_2^* \otimes (abe_1 + ae_2 - a^2 e_3) - e_3^* \otimes (be_1 + e_2 - ae_3),$$

$$r_5 = (ae_1 + e_3) \otimes e_2^* - e_2^* \otimes (ae_1 + e_3),$$

$$r_6 = e_1 \otimes e_2^* - e_2^* \otimes e_1,$$

$$r_7 = e_3 \otimes e_2^* - e_2^* \otimes e_3',$$

$$r_8 = 0,$$

$$r_9 = (ae_1 + be_3) \otimes e_2^* + e_1 \otimes e_3^* - e_2^* \otimes (ae_1 + be_3) - e_3^* \otimes e_1,$$

$$r_{10} = ae_1 \otimes e_2^* + e_1 \otimes e_3^* - e_2^* \otimes ae_1 - e_3^* \otimes e_1,$$

$$r_{11} = ae_3 \otimes e_2^* + e_1 \otimes e_3^* - e_2^* \otimes ae_3 - e_3^* \otimes e_1,$$

$$r_{12} = e_1 \otimes e_3^* - e_3^* \otimes e_1,$$

$$r_{13} = (ae_1 + e_2 + be_3) \otimes e_2^* - e_2^* \otimes (ae_1 + e_2 + be_3),$$

$$r_{14} = (ae_1 + e_2) \otimes e_2^* - e_2^* \otimes (ae_1 + e_2),$$

$$r_{15} = (e_2 + ae_3) \otimes e_2^* - e_2^* \otimes (e_2 + ae_3),$$

$$r_{16} = e_2 \otimes e_2^* - e_2^* \otimes e_2,$$

$$r_{17} = e_3 \otimes e_1^* + (ae_1 + be_3) \otimes e_2^* - e_3 \otimes e_3^* - e_1^* \otimes e_3 - e_2^* \otimes (ae_1 + be_3) + e_3^* \otimes e_3,$$

$$r_{18} = (ae_1 + e_3) \otimes e_1^* + be_3 \otimes e_2^* + ((a^2 - a) e_1 + (a-1) e_3) \otimes e_3^* - e_1^* \otimes (ae_1 + e_3) - e_2^* \otimes be_3 - e_3^* \otimes ((a^2 - a) e_1 + (a-1) e_3), \quad \left(a \neq \frac{1}{2}\right),$$

$$r_{19} = (ae_1 + e_3) \otimes e_1^* + be_1 \otimes e_2^* + ((a^2 - a) e_1 + (a-1) e_3) \otimes e_3^* - e_1^* \otimes (ae_1 + e_3) - e_2^* \otimes be_1 - e_3^* \otimes ((a^2 - a) e_1 + (a-1) e_3), \quad \left(a \neq \frac{1}{2}\right),$$

$$r_{20} = e_3 \otimes e_1^* + ae_3 \otimes e_2^* - e_3 \otimes e_3^* - e_1^* \otimes e_3 - e_2^* \otimes ae_3 + e_3^* \otimes e_3,$$

$$r_{21} = e_3 \otimes e_1^* + ae_1 \otimes e_2^* - e_3 \otimes e_3^* - e_1^* \otimes e_3 - e_2^* \otimes ae_1 + e_3^* \otimes e_3,$$

$$r_{22} = (ae_1 + e_3) \otimes e_1^* + \left(\left(a^2 - a\right) e_1 + (a-1) e_3\right) \otimes e_3^* - e_1^* \otimes (ae_1 + e_3) - e_3^* \otimes \left(\left(a^2 - a\right) e_1 + (a-1) e_3\right) \quad \left(a \neq \frac{1}{2}\right),$$

$$r_{23} = e_3 \otimes e_1^* - e_3 \otimes e_3^* - e_1^* \otimes e_3 + e_3^* \otimes e_3,$$

$$r_{24} = (ae_1 + e_3) \otimes e_1^* + (be_1 + ce_3) \otimes e_2^* + \left(\left(a^2 - a\right) e_1 + (a-1) e_3\right) \otimes e_3^* - e_1^* \otimes (ae_1 + e_3) - e_2^* \otimes (be_1 + ce_3) - e_3^* \otimes \left(\left(a^2 - a\right) e_1 + (a-1) e_3\right), \quad \left(a \neq \frac{1}{2}\right),$$

$$r_{25} = \left(\frac{1}{2}e_1 + e_3\right) \otimes e_1^* + ae_3 \otimes e_2^* - \left(\frac{1}{4}e_1 + \frac{1}{2}e_3\right) \otimes e_3^* - e_1^* \otimes \left(\frac{1}{2}e_1 + e_3\right) - e_2^* \otimes ae_3 + e_3^* \otimes \left(\frac{1}{4}e_1 + \frac{1}{2}e_3\right),$$

$$r_{26} = \left(\frac{1}{2}e_1 + e_3\right) \otimes e_1^* + ae_1 \otimes e_2^* - \left(\frac{1}{4}e_1 + \frac{1}{2}e_3\right) \otimes e_3^* - e_1^* \otimes \left(\frac{1}{2}e_1 + e_3\right) - e_2^* \otimes ae_1 + e_3^* \otimes \left(\frac{1}{4}e_1 + \frac{1}{2}e_3\right),$$

$$r_{27} = \left(\frac{1}{2}e_1 + e_3\right) \otimes e_1^* - \left(\frac{1}{4}e_1 + \frac{1}{2}e_3\right) \otimes e_3^* - e_1^* \otimes \left(\frac{1}{2}e_1 + e_3\right) + e_3^* \otimes \left(\frac{1}{4}e_1 + \frac{1}{2}e_3\right),$$

$$r_{28} = \left(\frac{1}{2}e_1 + e_3\right) \otimes e_1^* + (ae_1 + be_3) \otimes e_2^* - \left(\frac{1}{4}e_1 + \frac{1}{2}e_3\right) \otimes e_3^* - e_1^* \otimes \left(\frac{1}{2}e_1 + e_3\right) - e_2^* \otimes (ae_1 + be_3) + e_3^* \otimes \left(\frac{1}{4}e_1 + \frac{1}{2}e_3\right),$$

$$r_{29} = (e_2 + 2ae_3) \otimes e_1^* + \left(ae_2 + 2a^2 e_3\right) \otimes e_2^* - \left(\frac{1}{2}e_2 + ae_3\right) \otimes e_3^* - e_1^* \otimes (e_2 + 2ae_3) - e_2^* \otimes \left(ae_2 + 2a^2 e_3\right) + e_3^* \otimes \left(\frac{1}{2}e_2 + ae_3\right),$$

$$r_{30} = (ae_1 + e_2 + 2ae_3) \otimes e_1^* - \left(\frac{a}{2}e_1 + \frac{1}{2}e_2 + ae_3\right) \otimes e_3^* - e_1^* \otimes (ae_1 + e_2 + 2ae_3) + e_3^* \otimes \left(\frac{a}{2}e_1 + \frac{1}{2}e_2 + ae_3\right),$$

$$r_{31} = e_2 \otimes e_1^* - \frac{1}{2}e_2 \otimes e_3^* - e_1^* \otimes e_2 + e_3^* \otimes \frac{1}{2}e_2,$$

$$r_{32} = (ae_1 + e_2 + 2(a+b) e_3) \otimes e_1^* + \left(abe_1 + be_2 - 2\left(b^2 + ab\right) e_3\right) \otimes e_2^* - \left(\frac{a}{2}e_1 + \frac{1}{2}e_2 + (a+b) e_3\right) \otimes e_3^* - e_1^* \otimes (ae_1 + e_2 + 2(a+b) e_3) - e_2^* \otimes \left(abe_1 + be_2 - 2\left(b^2 + ab\right) e_3\right) + e_3^* \otimes \left(\frac{a}{2}e_1 + \frac{1}{2}e_2 + (a+b) e_3\right). \quad (30)$$

One can check that all of the tensors above are solutions of the classical Yang-Baxter equation in $g \ltimes_{\mathrm{ad}^*} g^*$.

4. Induced Left-Symmetric Algebras from Rota-Baxter Operators of Weight 0 on g

A left-symmetric algebra structure on g is a bilinear product $\cdot : g \otimes g \to g$ satisfying the condition

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z \quad (31)$$

for all $x, y, z \in g$. There are many examples of Lie algebras which do not admit a left-symmetric product. For example, it is easy to see that there are no left-symmetric algebras with semisimple Lie algebra. Equation (31) implies that the commutators $[x, y] = x \cdot y - y \cdot x$ satisfy the Jacobi identity; that is to say each left-symmetric product has an associated commutation Lie algebra, which is called the subadjacent Lie algebra. If R is a Rota-Baxter operator on a left-symmetric algebra, then R is a solution of CYBE on its subadjacent Lie algebra [17]. Clearly, each associative algebra product is a left-symmetric product. Given a Lie algebra g, it is a fundamental problem to decide whether g admits a left-symmetric product and to give a classification of such products [18]. As an application of Yang-Baxter operators, we can use them to construct left-symmetric algebras with respect to g_6.

Lemma 4 (see [13]). *Let g be a Lie algebra; P is called a solution of the classical Yang-Baxter equation. Define a new operation on g by*

$$x * y = [P(x), y], \quad \forall x, y \in g. \quad (32)$$

Then $(g, *)$ is a left-symmetric algebra.

According to Theorem 3 and Lemma 4, we can get some left-symmetric algebras of g.

Theorem 5. *Some left-symmetric algebras of g (of weight zero) are determined:*

(1) $e_3 * e_1 = -e_1$, $e_3 * e_2 = ae_1$, $e_3 * e_3 = e_1 + e_3$;

(2) $e_2 * e_1 = -ae_1$, $e_2 * e_2 = a^2(e_1 + e_3)$, $e_2 * e_3 = a(e_1 + e_3)$, $e_3 * e_1 = -e_1$, $e_3 * e_2 = a(e_1 + e_3)$, $e_3 * e_3 = e_1 + e_3$;

(3) $e_3 * e_1 = -e_1$, $e_3 * e_3 = e_1 + e_3$;

(4) $e_2 * e_1 = -ae_1$, $e_2 * e_2 = (a^2 + ab)e_1 + a^2 e_3$, $e_2 * e_3 = a(e_1 + e_3)$, $e_3 * e_1 = -e_1$, $e_3 * e_2 = (a + b)e_1 + ae_3$, $e_3 * e_3 = e_1 + e_3$;

(5) $e_2 * e_2 = (a - 1)e_1 - e_3$;

(6) $e_2 * e_2 = e_1$;

(7) $e_2 * e_2 = -(e_1 + e_3)$;

(8) $e_2 * e_2 = (a - b)e_1 - be_3$, $e_3 * e_2 = e_1$;

(9) $e_2 * e_2 = ae_1$, $e_3 * e_2 = e_1$;

(10) $e_2 * e_2 = -a(e_1 + e_3)$, $e_3 * e_2 = e_1$;

(11) $e_3 * e_2 = e_1$;

(12) $e_2 * e_1 = -e_1$, $e_2 * e_2 = (a - b)e_1 - be_3$, $e_2 * e_3 = e_1 + e_3$;

(13) $e_2 * e_1 = -e_1$, $e_2 * e_2 = ae_1$, $e_2 * e_3 = e_1 + e_3$;

(14) $e_2 * e_1 = -e_1$, $e_2 * e_2 = -a(e_1 + e_3)$, $e_2 * e_3 = e_1 + e_3$;

(15) $e_2 * e_1 = -e_1$, $e_2 * e_3 = e_1 + e_3$;

(16) $e_1 * e_2 = -(e_1 + e_3)$, $e_2 * e_2 = (a - b)e_1 - be_3$, $e_3 * e_2 = e_1 + e_3$;

(17) $e_1 * e_2 = (a - 1)e_1 - e_3$, $e_2 * e_2 = -b(e_1 + e_3)$, $e_3 * e_2 = (a - 1)^2 e_1 - (a - 1)e_3$;

(18) $e_1 * e_2 = (a - 1)e_1 - e_3$, $e_2 * e_2 = be_1$, $e_3 * e_2 = (a - 1)^2 e_1 - (a - 1)e_3$;

(19) $e_1 * e_2 = -(e_1 + e_3)$, $e_2 * e_2 = -a(e_1 + e_3)$, $e_3 * e_2 = e_1 + e_3$;

(20) $e_1 * e_2 = e_1 + e_3$, $e_2 * e_2 = ae_1$, $e_3 * e_2 = e_1 + e_3$;

(21) $e_1 * e_2 = (a - 1)e_1 - e_3$, $e_3 * e_2 = (a - 1)^2 e_1 - (a - 1)e_3$;

(22) $e_1 * e_2 = -(e_1 + e_3)$, $e_3 * e_2 = e_1 + e_3$;

(23) $e_1 * e_2 = (a - 1)e_1 - e_3$, $e_2 * e_2 = (b - c)e_1 - ce_3$, $e_3 * e_2 = (a - 1)^2 e_1 - (a - 1)e_3$;

(24) $e_1 * e_2 = -(1/2)e_1 - e_3$, $e_2 * e_2 = -a(e_1 + e_3)$, $e_3 * e_2 = (1/4)e_1 + (1/2)e_3$;

(25) $e_1 * e_2 = -(1/2)e_1 - e_3$, $e_2 * e_2 = ae_1$, $e_3 * e_2 = (1/4)e_1 + (1/2)e_3$;

(26) $e_1 * e_2 = -(1/2)e_1 - e_3$, $e_3 * e_2 = (1/4)e_1 + (1/2)e_3$;

(27) $e_1 * e_2 = -(1/2)e_1 - e_3$, $e_2 * e_2 = (a - b)e_1 - be_3$, $e_3 * e_2 = (1/4)e_1 + (1/2)e_3$;

(28) $e_1 * e_1 = -e_1$, $e_1 * e_2 = -2a(e_1 + e_3)$, $e_1 * e_3 = e_1 + e_3$, $e_2 * e_1 = -ae_1$, $e_2 * e_2 = -2a^2(e_1 + e_3)$, $e_2 * e_3 = a(e_1 + e_3)$, $e_3 * e_1 = (1/2)e_1$, $e_3 * e_2 = a(e_1 + e_3)$, $e_3 * e_3 = -(1/2)(e_1 + e_3)$;

(29) $e_1 * e_1 = -e_1$, $e_1 * e_2 = -ae_1 - 2ae_3$, $e_1 * e_3 = e_1 + e_3$, $e_3 * e_1 = (1/2)e_1$, $e_3 * e_2 = (a/2)e_1 + ae_3$, $e_3 * e_3 = -(1/2)(e_1 + e_3)$;

(30) $e_1 * e_3 = e_1 + e_3$, $e_3 * e_3 = -(1/2)(e_1 + e_3)$;

(31) $e_1 * e_1 = -e_1$, $e_1 * e_2 = -(a + 2b)e_1 - 2(a + b)e_3$, $e_1 * e_3 = e_1 + e_3$, $e_2 * e_1 = -be_1$, $e_2 * e_2 = (2b^2 + 3ab)e_1 + (2b^2 + 2ab)e_3$, $e_2 * e_3 = b(e_1 + e_3)$, $e_3 * e_1 = (1/2)e_1$, $e_3 * e_2 = (a/2 + b)e_1 + (a + b)e_3$, $e_3 * e_3 = -(1/2)(e_1 + e_3)$.

Conflicts of Interest

The authors declare no conflicts of interest.

Acknowledgments

This work is supported by the National Science Foundation of China (Grants nos. 11047030 and 11771122) and the Science and Technology Program of Henan Province (Grant no. 152300410061).

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