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Research Article

Relativistic Spinning Particle without Grassmann Variables and the Dirac Equation

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We present the relativistic particle model without Grassmann variables which, being canonically quantized, leads to the Dirac equation. Classical dynamics of the model is in correspondence with the dynamics of mean values of the corresponding operators in the Dirac theory. Classical equations for the spin tensor are the same as those of the Barut-Zanghi model of spinning particle.

1. Discussion: Nonrelativistic Spin

Starting from the classical works [1–6], a lot of efforts have been spent in attempts to understand behaviour of a particle with spin on the base of semiclassical mechanical models [7–24].

In the course of canonical quantization of a given classical theory, one associates Hermitian operators with classical variables. Let z^α stands for the basic phase-space variables that describe the classical system, and $\{z^\alpha, z^\beta\}$ is the corresponding classical bracket. (It is the Poisson (Dirac) bracket in the theory without (with) second-class constraints.) According to the Dirac quantization paradigm [3], the operators $\hat{z}^\alpha$ must be chosen to obey the quantization rule

$$\left[\hat{z}^\alpha, \hat{z}^\beta\right]\mp = i\hbar \left\{z^\alpha, z^\beta\right\}\Big|_{z\to\hat{z}}. \tag{1.1}$$

In this equation, we take commutator (anticommutator) of the operators for the antisymmetric (symmetric) classical bracket. Antisymmetric (symmetric) classical bracket arises in the classical mechanics of even (odd = Grassmann) variables.

Since the quantum theory of spin is known (it is given by the Pauli (Dirac) equation for nonrelativistic (relativistic) case), search for the corresponding semiclassical model represents the inverse task to those of canonical quantization: we look for the classical-mechanics system whose classical bracket obeys (1.1) for the known left-hand side. Components of the nonrelativistic spin operator $\widehat{S}_i = (\hbar/2)\sigma^i$ (σ^i are the Pauli matrices (1.5)) form a simple algebra with respect to commutator

$$\left[\widehat{S}_i, \widehat{S}_j\right]_- = i\hbar\epsilon_{ijk}\widehat{S}_k, \tag{1.2}$$

as well as to anticommutator

$$\left[\widehat{S}_i, \widehat{S}_j\right]_+ = \frac{\hbar^2}{2}\delta_{ij}. \tag{1.3}$$

So, the operators can be produced starting from a classical model based on either even or odd spin-space variables.

In their pioneer work [13, 14], Berezin and Marinov have constructed the model based on the odd variables and showed that it gives very economic scheme for semi-classical description of both nonrelativistic and relativistic spin. Their prescription can be shortly resumed as follows. For nonrelativistic spin, the noninteracting Lagrangian reads $(m/2)(\dot{x}_i)^2 + (i/2)\xi_i\dot{\xi}_i$, where the spin inner space is constructed from vector-like Grassmann variables ξ_i, $\xi_i\xi_j = -\xi_j\xi_i$. Since the Lagrangian is linear on $\dot{\xi}_i$, their conjugate momenta coincide with ξ, $\pi_i = \partial L/\partial\dot{\xi}_i = i\xi_i$. The relations represent the Dirac second-class constraints and are taken into account by transition from the odd Poisson bracket to the Dirac one, the latter reads

$$\left\{\xi_i, \xi_j\right\}_{\mathrm{DB}} = i\delta_{ij}. \tag{1.4}$$

Dealing with the Dirac bracket, one can resolve the constraints, excluding the momenta from consideration. So, there are only three spin variables ξ_i with the desired brackets (1.4). According to (1.1), (1.4), and (1.3), canonical quantization is performed replacing the variables by the spin operators $\widehat{S}_i$ proportional to the Pauli σ-matrices, $\widehat{S}_i = (\hbar/2)\sigma^i$, $[\sigma^i, \sigma^j]_+ = 2\delta^{ij}$,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.5}$$

acting on two-dimensional spinor space $\Psi_\alpha(t, \vec{x})$. Canonical quantization of the particle on an external electromagnetic background leads to the Pauli equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(\frac{1}{2m}\left(\widehat{p}_i - \frac{e}{c}A_i\right)^2 - eA_0 - \frac{e\hbar}{2mc}B_i\sigma^i\right)\Psi. \tag{1.6}$$

It has been denoted that $p_i = -i\hbar(\partial/\partial x_i)$, (A_0, A_i) is the four-vector potential of electromagnetic field, and the magnetic field is $B_i = \epsilon_{ijk}\partial_j A_k$, where ϵ_{ijk} represents the totally antisymmetric tensor with $\epsilon_{123} = 1$. Relativistic spin is described in a similar way [13, 14].

The problem here is that the Grassmann classical mechanics represents a rather formal mathematical construction. It leads to certain difficulties [13, 14, 17] in attempts to use it for description the spin effects on the semiclassical level, before the quantization. Hence it would be interesting to describe spin on a base of usual variables. While the problem has a long history (see [7–15] and references therein), there appears to be no wholly satisfactory solution to date. It seems to be surprisingly difficult [15] to construct, in a systematic way, a consistent model that would lead to the Dirac equation in the the course of canonical quantization. It is the aim of this work to construct an example of mechanical model for the Dirac equation.

To describe the nonrelativistic spin by commuting variables, we need to construct a mechanical model which implies the commutator (even) operator algebra (1.2) instead of the anticommutator one (1.3). It has been achieved in the recent work [18] starting from the Lagrangian

$$L = \frac{m}{2}(\dot{x}_i)^2 + \frac{e}{c}A_i\dot{x}_i + eA_0 + \frac{1}{2g}\left(\dot{\omega}_i - \frac{e}{mc}\epsilon_{ijk}\omega_j B_k\right)^2 + \frac{3g\hbar^2}{8a^2} + \frac{1}{\phi}\left((\omega_i)^2 - a^2\right). \tag{1.7}$$

The configuration-space variables are $x_i(t)$, $\omega_i(t)$, $g(t)$, $\phi(t)$. Here x_i represents the spatial coordinates of the particle with the mass m and the charge e, ω_i are the spin-space coordinates, g, ϕ are the auxiliary variables and a = const. Second and third terms in (1.7) represent minimal interaction with the vector potential A_0, A_i of an external electromagnetic field, while the fourth term contains interaction of spin with a magnetic field. At the end, it produces the Pauli term in quantum mechanical Hamiltonian.

The Dirac constraints presented in the model imply [18] that spin lives on two-dimensional surface of six-dimensional spin phase space ω_i, π_i. The surface can be parameterized by the angular-momentum coordinates $S_i = \epsilon_{ijk}\omega_j\pi_k$, subject to the condition $S^2 = 3\hbar^2/4$. They obey the classical brackets $\{S_i, S_j\} = \epsilon_{ijk}S_k$. Hence we quantize them according the rule $S_i \to \widehat{S}_i$.

The model leads to reasonable picture both on classical and quantum levels. The classical dynamics is governed by the Lagrangian equations

$$m\ddot{x}_i = eE_i + \frac{e}{c}\epsilon_{ijk}\dot{x}_j B_k - \frac{e}{mc}S_k\partial_i B_k, \tag{1.8}$$

$$\dot{S}_i = \frac{e}{mc}\epsilon_{ijk}S_j B_k. \tag{1.9}$$

It has been denoted that $E = -(1/c)(\partial A/\partial t) + \nabla A_0$. Since $S^2 \approx \hbar^2$, the S-term disappears from (1.8) in the classical limit $\hbar \to 0$. Then (1.8) reproduces the classical motion on an external electromagnetic field. Notice also that in absence of interaction, the spinning particle does not experience an undesirable *Zitterbewegung*. Equation (1.9) describes the classical spin precession in an external magnetic field. On the other hand, canonical quantization of the model immediately produces the Pauli equation (1.6).

Below, we generalize this scheme to the relativistic case, taking angular-momentum variables as the basic coordinates of the spin space. On this base, we construct the relativistic-invariant classical mechanics that produces the Dirac equation after the canonical quantization, and briefly discuss its classical dynamics.

2. Algebraic Construction of the Relativistic Spin Space

We start from the model-independent construction of the relativistic-spin space. Relativistic equation for the spin precession can be obtained including the three-dimensional spin vector $\widehat{S}^i$ (1.2) either into the Frenkel tensor $\Phi^{\mu\nu}$, $\Phi^{\mu\nu}u_\nu = 0$, or into the Bargmann-Michel-Telegdi four-vector (The conditions $\Phi^{\mu\nu}u_\nu = 0$ and $S^\mu u_\mu = 0$ guarantee that in the rest frame survive only three components of these quantities, which implies the right nonrelativistic limit). S^μ, $S^\mu u_\mu = 0$, where u_μ represents four-velocity of the particle. Unfortunately, the semiclassical models based on these schemes do not lead to a reasonable quantum theory, as they do not produce the Dirac equation through the canonical quantization. We now motivate that it can be achieved in the formulation that implies inclusion of $\widehat{S}_i$ into the $SO(3,2)$ angular-momentum tensor $\widehat{L}^{AB}$ of five-dimensional Minkowski space $A = (\mu, 5) = (0, i, 5) = (0, 1, 2, 3, 5)$, $\eta^{AB} = (- + + + -)$.

In the passage from nonrelativistic to relativistic spin, we replace the Pauli equation by the Dirac one

$$\left(\widehat{p}_\mu \Gamma^\mu + mc\right)\Psi(x^\mu) = 0, \tag{2.1}$$

where $\widehat{p}_\mu = -i\hbar\partial_\mu$. We use the representation with Hermitian Γ^0 and anti-Hermitian Γ^i

$$\Gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \Gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{2.2}$$

then $[\Gamma^\mu, \Gamma^\nu]_+ = -2\eta^{\mu\nu}$, $\eta^{\mu\nu} = (- + + +)$, and $\Gamma^0\Gamma^i$, Γ^0 are the Dirac matrices [3] α^i, β

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

We take the classical counterparts of the operators $\widehat{x}^\mu$ and $\widehat{p}_\mu = -i\hbar\partial_\mu$ in the standard way, which are x^μ, p^ν, with the Poisson brackets $\{x^\mu, p^\nu\}_{\mathrm{PB}} = \eta^{\mu\nu}$.

Let us discuss the classical variables that could produce the Γ-matrices. To this aim, we first study their commutators. The commutators of Γ^μ do not form closed Lie algebra, but produce SO(1,3)-Lorentz generators

$$[\Gamma^\mu, \Gamma^\nu]_- = -2i\Gamma^{\mu\nu}, \tag{2.4}$$

where it has been denoted $\Gamma^{\mu\nu} \equiv (i/2)(\Gamma^\mu\Gamma^\nu - \Gamma^\nu\Gamma^\mu)$. The set Γ^μ, $\Gamma^{\mu\nu}$ form closed algebra. Besides the commutator (2.4), one has

$$\begin{gathered} [\Gamma^{\mu\nu}, \Gamma^\alpha]_- = 2i\left(\eta^{\mu\alpha}\Gamma^\nu - \eta^{\nu\alpha}\Gamma^\mu\right), \\ \left[\Gamma^{\mu\nu}, \Gamma^{\alpha\beta}\right]_- = 2i\left(\eta^{\mu\alpha}\Gamma^{\nu\beta} - \eta^{\mu\beta}\Gamma^{\nu\alpha} - \eta^{\nu\alpha}\Gamma^{\mu\beta} + \eta^{\nu\beta}\Gamma^{\mu\alpha}\right). \end{gathered} \tag{2.5}$$

The algebra (2.4), (2.5) can be identified with the five-dimensional Lorentz algebra $SO(2,3)$ with generators $\widehat{L}^{AB}$

$$\left[\widehat{L}^{AB}, \widehat{L}^{CD}\right]_{-} = 2i\left(\eta^{AC}\widehat{L}^{BD} - \eta^{AD}\widehat{L}^{BC} - \eta^{BC}\widehat{L}^{AD} + \eta^{BD}\widehat{L}^{AC}\right), \tag{2.6}$$

assuming $\Gamma^{\mu} \equiv \widehat{L}^{5\mu}$, $\Gamma^{\mu\nu} \equiv \widehat{L}^{\mu\nu}$.

To reach the algebra starting from a classical-mechanics model, we introduce ten-dimensional "phase" space of the spin degrees of freedom, ω^A, π^B, equipped with the Poisson bracket

$$\left\{\omega^A, \pi^B\right\}_{\mathrm{PB}} = \eta^{AB}. \tag{2.7}$$

Then Poisson brackets of the quantities

$$J^{AB} \equiv 2\left(\omega^A\pi^B - \omega^B\pi^A\right) \tag{2.8}$$

read

$$\left\{J^{AB}, J^{CD}\right\}_{\mathrm{PB}} = 2\left(\eta^{AC}J^{BD} - \eta^{AD}J^{BC} - \eta^{BC}J^{AD} + \eta^{BD}J^{AC}\right). \tag{2.9}$$

Below we use the decompositions

$$J^{AB} = \left(J^{5\mu}, J^{\mu\nu}\right) = \left(J^{50},\ J^{5i} \equiv \mathbf{J}^5,\ J^{0i} \equiv \mathbf{W},\ J^{ij} = \epsilon^{ijk}D^k\right). \tag{2.10}$$

The Jacobian of the transformation $(\omega^A, \pi^B) \rightarrow J^{AB}$ has rank equal seven. So, only seven among ten functions $J^{AB}(\omega, \pi)$, $A < B$, are independent quantities. They can be separated as follows. By construction, the quantities (2.8) obey the identity

$$\epsilon^{\mu\nu\alpha\beta}J^5{}_{\nu}J_{\alpha\beta} = 0, \quad \Longleftrightarrow J^{ij} = \frac{1}{J^{50}}\left(J^{5i}J^{0j} - J^{5j}J^{0i}\right), \tag{2.11}$$

that is, the three-vector $\mathbf{D}$ can be presented through $\mathbf{J}^5$, $\mathbf{W}$ as

$$\mathbf{D} = \frac{1}{J^{50}}\mathbf{J}^5 \times \mathbf{W}. \tag{2.12}$$

Further, (ω^A, π^B)-space can be parameterized by the coordinates $J^{5\mu}$, J^{0i}, ω^0, ω^5, π^5. We can not yet quantize the variables since it would lead to the appearance of some operators $\widehat{\omega}^0$, $\widehat{\omega}^5$, $\widehat{\pi}^5$, which are not presented in the Dirac theory and are not necessary for description of spin. To avoid the problem, we kill the variables ω^0, ω^5, π^5, restricting our model to live on seven-dimensional surface of ten-dimensional phase space ω^A, π^B. The only $SO(2,3)$

quadratic invariants that can be constructed from ω^A, π^B are $\omega^A\omega_A, \omega^A\pi_A, \pi^A\pi_A$. We choose conventionally the surface determined by the equations

$$\omega^A\omega_A + R = 0, \tag{2.13}$$

$$\omega^A\pi_A = 0, \qquad \pi^A\pi_A = 0, \tag{2.14}$$

$R = \text{const} > 0$. The quantities $J^{5\mu}$, J^{0i} form a coordinate system of the spin-space surface. So, we can quantize them instead of the initial variables ω^A, π^B.

According to (1.1), (2.6), (2.9), quantization is achieved replacing the classical variables $J^{5\mu}$, $J^{\mu\nu}$ on Γ-matrices

$$J^{5\mu} \longrightarrow \hbar\Gamma^\mu, \qquad J^{\mu\nu} \longrightarrow \hbar\Gamma^{\mu\nu}. \tag{2.15}$$

It implies, that the Dirac equation can be produced by the constraint (we restate that $J^{5\mu} \equiv 2(\omega^5\pi^\mu - \omega^\mu\pi^5)$)

$$p_\mu J^{5\mu} + mc\hbar = 0. \tag{2.16}$$

Summing up, to describe the relativistic spin, we need a theory that implies the Dirac constraints (2.13), (2.14), (2.16) in the Hamiltonian formulation.

3. Dynamical Realization

One possible dynamical realization of the construction presented above is given by the following $d = 4$ Poincare-invariant Lagrangian

$$L = -\frac{1}{2e_2}\left[(\dot{x}^\mu + e_3\omega^\mu)^2 - \left(e_3\omega^5\right)^2\right] - \frac{\sigma mc\hbar}{2\omega^5} + \frac{1}{\sigma}\left[(\dot{x}^\mu + e_3\omega^\mu)\dot{\omega}_\mu - e_3\omega^5\dot{\omega}^5\right] - e_4\left(\omega^A\omega_A + R\right), \tag{3.1}$$

written on the configuration space $x^\mu, \omega^\mu, \omega^5, e_i, \sigma$, where e_i, σ are the auxiliary variables. The variables ω^5, e_i, σ are scalars under the Poincare transformations. The remaining variables transform according to the rule

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \qquad \omega'^\mu = \Lambda^\mu{}_\nu\omega^\nu. \tag{3.2}$$

Local symmetries of the theory form the two-parameter group composed by the reparametrizations

$$\delta x^\mu = \alpha\dot{x}^\mu, \qquad \delta\omega^A = \alpha\dot{\omega}^A, \qquad \delta e_i = (\alpha e_i)^{\cdot}, \qquad \delta\sigma = (\alpha\sigma)^{\cdot}, \tag{3.3}$$

as well as by the local transformations with the parameter $\epsilon(\tau)$ (below we have denoted $\beta \equiv \dot{e}_4\epsilon + (1/2)e_4\dot{\epsilon}$)

$$\delta x^\mu = 0, \qquad \delta\omega^A = \beta\omega^A, \qquad \delta\sigma = \beta\sigma, \quad \delta e_2 = 0,$$
$$\delta e_3 = -\beta e_3 + \frac{e_2}{\sigma}\dot{\beta}, \qquad \delta e_4 = -2e_4\beta - \left(\frac{e_2\dot{\beta}}{2\sigma^2}\right)^{\cdot}. \tag{3.4}$$

The local symmetries guarantee appearance of the first-class constraints (2.13), (2.16).

Curiously enough, the action can be rewritten in almost five-dimensional form. Indeed, after the change $(x^\mu, \sigma, e_3) \to (\tilde{x}^\mu, \tilde{x}^5, \tilde{e}_3)$, where $\tilde{x}^\mu = x^\mu - (e_2/\sigma)\omega^\mu$, $\tilde{x}^5 = -(e_2/\sigma)\omega^5$, $\tilde{e}_3 = e_3 + (e_2/\sigma)$, it reads

$$L = -\frac{1}{2e_2}\left(D\tilde{x}^A\right)^2 + \frac{\left(\tilde{x}^5\right)^2}{2e_2(\omega^5)^2}\left(\dot{\omega}^A\right)^2 + \frac{e_2 mc\hbar}{\tilde{x}^5} - e_4\left(\omega^A\omega_A + R\right), \tag{3.5}$$

where the covariant derivative is $D\tilde{x}^A = \dot{\tilde{x}}^A + \tilde{e}_3\omega^A$ (The change is an example of conversion of the second-class constraints in the Lagrangian formulation [25]).

Canonical Quantization

In the Hamiltonian formalism, the action implies the desired constraints (2.13), (2.14), (2.16). The constraints (2.13), (2.14) can be taken into account by transition from the Poisson to the Dirac bracket, and after that they are omitted from the consideration [17, 26]. The first-class constraint (2.16) is imposed on the state vector and produces the Dirac equation. In the result, canonical quantization of the model leads to the desired quantum picture.

We now discuss some properties of the classical theory and confirm that they are in correspondence with semiclassical limit [3, 27, 28] of the Dirac equation.

Equations of Motion

The auxiliary variables e_i, σ can be omitted from consideration after the partial fixation of a gauge. After that, Hamiltonian of the model reads

$$H = \frac{1}{2}\pi^A\pi_A + \frac{1}{2}\left(p_\mu J^{5\mu} + mc\hbar\right). \tag{3.6}$$

Since ω^A, π^A are not ϵ-invariant variables, their equations of motion have no much sense. So, we write the equations of motion for ϵ-invariant quantities x^μ, p_μ, $J^{5\mu}$, $J^{\mu\nu}$

$$\dot{x}^\mu = \frac{1}{2}J^{5\mu}, \tag{3.7}$$

$$\dot{J}^{5\mu} = -J^{\mu\nu}p_\nu,$$
$$\dot{J}^{\mu\nu} = p^\mu J^{5\nu} - p^\nu J^{5\mu}, \tag{3.8}$$

$$\dot{p}^\mu = 0. \tag{3.9}$$

They imply

$$\ddot{x}^{\mu} = -\frac{1}{2}J^{\mu\nu}p_{\nu}. \tag{3.10}$$

In three-dimensional notations, the equation (3.8) read

$$\dot{J}^{50} = -(\mathbf{W}\mathbf{p}), \qquad \dot{\mathbf{J}}^{5} = -p^{0}\mathbf{W} + \mathbf{D}\times\mathbf{p}, \\ \dot{\mathbf{W}} = p^{0}\mathbf{J}^{5} - J^{50}\mathbf{p}. \tag{3.11}$$

Relativistic Invariance

While the canonical momentum of x^{μ} is given by p^{μ}, the mechanical momentum, according to (3.7), coincides with the variables that turn into the Γ-matrices in quantum theory, $(1/2)J^{5\mu}$. Due to the constraints (2.13), (2.14), $J^{5\mu}$ obeys $(J^{5\mu}/\pi^{5})^{2} = -4R$, which is analogy of $p^{2} = -m^{2}c^{2}$ of the spinless particle. As a consequence, $x^{i}(t)$ cannot exceed the speed of light, $(dx^{i}/dt)^{2} = c^{2}(\dot{x}^{i}/\dot{x}^{0})^{2} = c^{2}(1-((\pi^{5})^{2}4R/(J^{50})^{2})) < c^{2}$. Equations (3.10), (3.11) mean that both x^{μ}-particle and the variables $\mathbf{W}$, $\mathbf{J^5}$ experience the *Zitterbewegung* in noninteracting theory.

Center-of-Charge Rest Frame

Identifying the variables x^{μ} with position of the charge, (3.7) implies that the rest frame is characterized by the conditions

$$J^{50} = \text{const}, \qquad \mathbf{J}^{5} = 0. \tag{3.12}$$

According to (3.12), (2.12), only $\mathbf{W}$ survives in the nonrelativistic limit.

The Variables Free of Zitterbewegung

The quantity (center-of-mass coordinate [7]) $\tilde{x}^{\mu} = x^{\mu} + (1/2p^{2})J^{\mu\nu}p_{\nu}$ obeys $\dot{\tilde{x}}^{\mu} = -(mc\hbar/2p^{2})p^{\mu}$, so, it has the free dynamics $\ddot{\tilde{x}}^{\mu} = 0$. Note also that p^{μ} represents the mechanical momentum of $\tilde{x}^{\mu}$-particle.

As the classical four-dimensional spin vector, let us take $S^{\mu} = \epsilon^{\mu\nu\alpha\beta}p_{\nu}J_{\alpha\beta}$. It has no precession in the free theory, $\dot{S}^{\mu} = 0$. In the rest frame, it reduces to $\mathbf{S}^{0} = 0$, $\mathbf{S} = \mathbf{p}\times\mathbf{W}$.

Comparison with the Barut-Zanghi (BZ) Model

The BZ spinning particle [16] is widely used [19–24] for semiclassical analysis of spin effects. Starting from the even variable z_{α}, where $\alpha = 1,2,3,4$ is SO(1,3)-spinor index, Barut and Zanghi have constructed the spin-tensor according to $S_{\mu\nu} = (1/4)i\bar{z}\gamma_{\mu\nu}z$. We point out that (3.7)–(3.9) of our model coincide with those of BZ-model, identifying $J^{5\mu} \leftrightarrow v^{\mu}$, $J^{\mu\nu} \leftrightarrow S^{\mu\nu}$. Besides, our model implies the equations $(J^{5\mu}/\pi^{5})^{2} = -4R, p_{\mu}J^{5\mu} + mc\hbar = 0$. The first equation guarantees that the center of charge cannot exceed the speed of light. The second equation implies the Dirac equation. (In the BZ theory [16], the mass of the spinning particle is not fixed from the model.)

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