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*Research Article*

# On the Solution of a Hyperbolic One-Dimensional Free Boundary Problem for a Maxwell Fluid

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We study a hyperbolic (telegrapher's equation) free boundary problem describing the pressure-driven channel flow of a Bingham-type fluid whose constitutive model was derived in the work of Fusi and Farina (2011). The free boundary is the surface that separates the inner core (where the velocity is uniform) from the external layer where the fluid behaves as an upper convected Maxwell fluid. We present a procedure to obtain an explicit representation formula for the solution. We then exploit such a representation to write the free boundary equation in terms of the initial and boundary data only. We also perform an asymptotic expansion in terms of a parameter tied to the rheological properties of the Maxwell fluid. Explicit formulas of the solutions for the various order of approximation are provided.

## 1. Introduction

In this paper we study the well posedness of a hyperbolic free boundary problem arisen from a one-dimensional model for the channel flow of a rate-type fluid with stress threshold presented in [1]. The model describes the one-dimensional flow of a fluid which behaves as a nonlinear viscoelastic fluid if the stress is above a certain threshold $\tau_o$ and like a rate type fluid if the stress is below that threshold. The problem investigated here belongs to a series of extensions of the classical Bingham model we have proposed in recent years (see [2–5]).

In particular, in [1] we describe the one-dimensional flow of such a fluid in an infinite channel, assuming that in the outer part of the channel the material behaves as a viscoelastic upper convected Maxwell fluid, while in the inner core as a rate-type Oldroyd-B fluid. The general mathematical model is derived within the framework of the theory of natural configurations developed by Rajagopal and Srinivasa (see [6]). The constitutive equations are obtained imposing how the system stores and dissipates energy and exploiting the criterion of the maximization of the dissipation rate.

The main practical motivation behind this study comes from the analysis of materials like asphalt or bitumen which exhibit a stress threshold beyond which they change its rheological properties. Indeed from the papers [7–9], it is clear that such materials have a viscoelastic behaviour (for instance, upper convected Maxwell fluid) which is observed if the applied stress is greater than a certain threshold (see, in particular, [7]).

The mathematical formulation for the channel flow driven by a constant pressure gradient consists in a free boundary problem involving a hyperbolic telegrapher's equation (Maxwell fluid) and a third-order equation (Oldroyd-B fluid). The free boundary is the surface dividing the two domains: the inner channel core and the external layer. Due to the high complexity of the general problem, here we have considered a simplified version which arises when the order of magnitude of some physical parameters involved in the general model ranges around particular values. In such a case we have that the velocity of the inner core is constant in space and time, while the outer part behaves as a viscoelastic upper convected Maxwell fluid (see [1] for more details). The mathematical formulation turns out to be a hyperbolic free boundary problem which, in the authors knowledge, is new since it involves a telegrapher's equation coupled with an ODE describing the evolution of the interface.

The paper is structured as follows. In Section 2 we formulate the problem, namely, problem (2.1), and specify the basic assumptions. In Section 3 we give an equivalent formulation of the problem which leads to a nonlinear integrodifferential equation for the free boundary. We prove local existence and uniqueness for such an equation (see Theorem 3.3), under specific assumptions on the data.

The interesting aspect of the mathematical analysis lies on the technique we employ to reduce the complete problem to a single integrodifferential equation from which some mathematical properties can be derived (the free boundary equation can be solved autonomously from the governing equation of the velocity field). Such a methodology is a generalization of a technique already introduced in [2].

In Section 4 we perform an asymptotic expansion in terms of a coefficient $\omega$ (representing the ration between the elastic characteristic time and the relaxation time of the viscoelastic material), which typically is of the order $O(10^{-1})$. This procedure allows to obtain approximations of the actual solution up to any order through an iterative procedure. We do not prove the convergence of the asymptotic approximations to the actual solution (whose existence is proved in Theorem 3.3), limiting ourselves to develop only the formal procedure. Indeed, the main advantage of this procedure is that, for each order of approximation, the governing equation for the velocity field is the "standard" wave equation, which is by far easier to handle than the telegrapher's equation. We end the paper with few conclusive remarks.

## 2. Mathematical Formulation

In this section we state the mathematical problem. We refer the reader to [1] for all the details describing how this simplified model was derived from the general one. In the general case, in the region $[0, s]$, the fluid behaves as an Oldroyd-B type fluid. The problem we are studying here is a particular case of such a model, which stems from some specific assumptions on the physical parameters (fulfilled by materials like asphalt and bitumen). Under such assumptions, the inner core $[0, s]$ moves with uniform constant velocity $V_o$.

We consider an orthogonal coordinate system $xoy$ and assume that the fluid is confined between two parallel plates placed at distance $2L$. We assume that the motion takes place along the $x$-direction and that the velocity field has the form $\vec{v}(y, \mathrm{t}) = v(y, t)\vec{i}$. We rescale the problem (in a nondimensional form) and, because of symmetry, we study the upper part

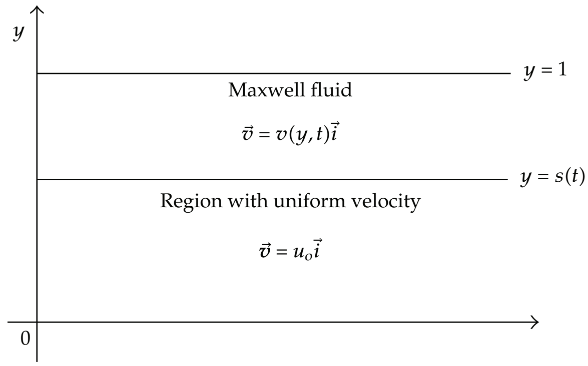


**Figure 1:** Upper part of the channel.

of the layer $y \in [0,1]$ (the space variable is rescaled by $L$). The geometry of the system we investigate is depicted in Figure 1.

The mathematical model is written for the velocity field $v(y,t)$ in the viscoelastic region which is separated from the region with zero strain rate (uniform velocity) by the moving interface $y = s(t)$.

The nondimensional formulation is the following:

$$\begin{aligned}
&v_{tt} + 2\omega v_t = v_{yy} + \beta^2 \quad y \in (s,1),\ t > 0,\\
&v(y,0) = v_o(y) \quad y \in (s_o,1),\\
&v_t(y,0) = 0 \quad y \in (s_o,1),\\
&v(1,t) = 0 \quad t > 0,\\
&v(s,t) = V_o \quad t > 0,\\
&v_y(s,t) + \dot{s}v_t(s,t) = -\beta^2 \quad \mathtt{Bn} t > 0,\\
&s(0) = s_o, \quad s_o \in (0,1).
\end{aligned} \tag{2.1}$$

where

(i) $\rho$ is the material density,

(ii) $\eta$ is the viscosity of the fluid,

(iii) $\mu$ is the elastic modulus,

(iv) $\beta^2$ is a positive parameter depending on the viscosity $\eta$ (see [1]),

(v) Bn is the Bingham number,

(vi) $V_o$ is the velocity of the inner core,

(vii) $2\omega = t_e/t_r$,

(viii) $t_e = L\sqrt{\rho/\mu}$ is the characteristic elastic time,

(ix) $t_r = \eta/2\mu$ is the relaxation time.

In the case of asphalt typical values are (see [8, 9])

$$\mu = 1\,\text{MPa}, \qquad \rho = 1.5 \times 10^3\,\text{Kg/m}^3, \qquad \eta = 10^2\,\text{MPa}\cdot\text{s}. \tag{2.2}$$

Taking $L = 500\,\mathrm{m}$ we get

$$t_e = 15\,\mathrm{s}, \qquad t_r = 50\,\mathrm{s}, \quad \Longrightarrow \omega = 0.15. \tag{2.3}$$

*Remark 2.1.* In [1] we have proved that problem (2.1) admits a stationary solution provided

$$V_o \leqslant \beta^2\left(\frac{1}{2} + \mathrm{Bn}\right) \tag{2.4}$$

and that the stationary solution is given by

$$\begin{gathered} v_\infty(y) = -\frac{\beta^2}{2}(s - \mathrm{Bn} - y)^2 + \frac{\beta^2 \mathrm{Bn}}{2} + V_o, \\ s_\infty = 1 + \mathrm{Bn} - \sqrt{\mathrm{Bn}^2 + \frac{2V_o}{\beta^2}}. \end{gathered} \tag{2.5}$$

## 3. An Equivalent Formulation

Before proceeding in proving analytical results of problem (2.1) we introduce the new coordinate system

$$x = 1 - y, \quad \Longleftrightarrow y = 1 - x, \tag{3.1}$$

and the new variable

$$U(x,t) = \exp(\omega t)v(1-x,t), \quad \Longleftrightarrow v(y,t) = U(1-y)\exp(-\omega t). \tag{3.2}$$

With transformations (3.1)-(3.2), problem (2.1) becomes

$$\begin{gathered} U_{xx} - U_{tt} + \omega^2 U = -\beta^2 \exp(\omega t), \quad x \in (0,\xi),\ t > 0, \\ U(x,0) = U_o(x), \quad x \in (0,\xi_o), \\ U_t(x,0) = U_1(x), \quad x \in (0,\xi_o), \\ U(0,t) = 0, \quad t > 0, \\ U(\xi,t) = \exp(\omega t)V_o, \quad t > 0, \\ U_x(\xi,t) + \dot{\xi}U_t(\xi,t) - \dot{\xi}\omega U(\xi,t) = \exp(\omega t)\beta^2 \mathrm{Bn}, \quad t > 0, \\ \xi(0) = \xi_o, \quad \xi_o \in (0,1), \end{gathered} \tag{3.3}$$

where

$$\xi(t) = 1 - s(t), \quad \xi_o = 1 - s_o, \quad U_o(x) = v_o(1-x), \quad U_1(x) = \omega U_o(x). \tag{3.4}$$

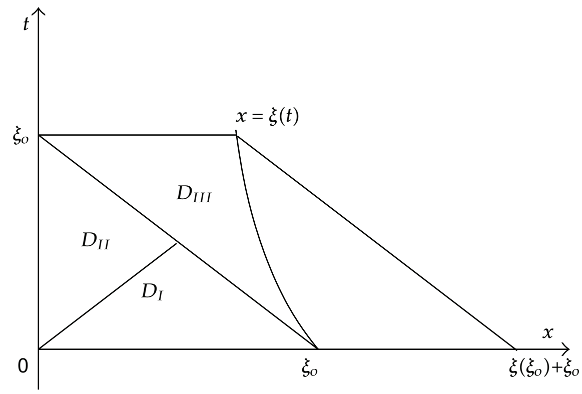


**Figure 2:** Sketch of the domain.

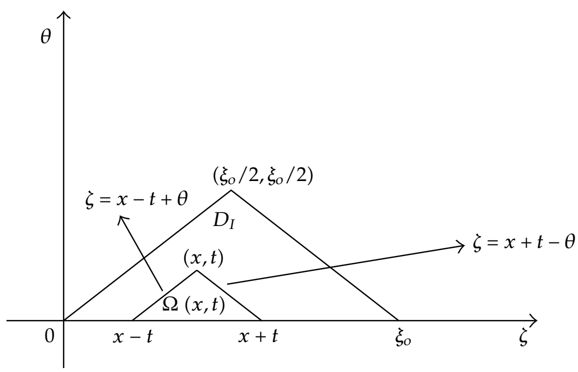


**Figure 3:** The domain $D_I$.

Notice that, by means of (3.2), the evolution equation for the new variable $U(x,t)$ has become a nonhomogeneous Klein-Gordon equation [10].

The domain of problem (3.3) is depicted in Figure 2. We begin by considering the domain $D_I$ (see Figure 3). Here the solution has the representation formula (see [11])

$$U(x,t) = \frac{1}{2}[U_o(x-t) + U_o(x+t)] + \frac{1}{2}\int_{x-t}^{x+t}[R(x,t;\zeta,0)U_1(\zeta) - R_\theta(x,t;\zeta,0)U_o(\zeta)]d\zeta \\ + \frac{\beta^2}{2}\int_0^t \exp(\omega\theta)d\theta \int_{x-t+\theta}^{x+t-\theta} R(x,t;\zeta,\theta)d\zeta, \tag{3.5}$$

where $R(x,t;\zeta,\theta)$ is the Riemann's function that solves the problem (see again Figure 3)

$$\begin{gathered} R_{\zeta\zeta} - R_{\theta\theta} + \omega^2 R = 0 \quad (\zeta,\theta) \in \Omega(x,t), \\ R(x,t;x+t-\theta,\theta) = 1 \quad \theta \in [0,t], \\ R(x,t;x-t+\theta,\theta) = 1 \quad \theta \in [0,t], \\ \Omega(x,t) = \{(x,t): \ x-t+\theta \leqslant \zeta \leqslant x+t-\theta, \ 0 \leqslant \theta \leqslant t\}. \end{gathered} \tag{3.6}$$

To determine the solution of problem (3.6) we set

$$z = \sqrt{(t-\theta)^2 - (\zeta - x)^2}, \tag{3.7}$$

where

$$\begin{aligned} z_\theta^2 - z_\zeta^2 &= 1, \\ z_{\theta\theta} - z_{\zeta\zeta} &= \frac{1}{z}. \end{aligned} \tag{3.8}$$

By means of (3.7) problem (3.6) becomes

$$\begin{gathered} R''(z) + \frac{R'(z)}{z} - \omega^2 R(z) = 0 \\ R(0) = 1, \end{gathered} \tag{3.9}$$

where $(3.9)_{(1)}$ is the modified Bessel equation of zero order. The solution of (3.9) is given by

$$R(z) = R(x,t;\zeta,\theta) = I_o\left(\omega\sqrt{(t-\theta)^2 - (\zeta - x)^2}\right), \tag{3.10}$$

where $I_o$ is the modified Bessel function of zero order. It is easy to prove that the function defined by (3.10) satisfies problem (3.6). Moreover, since [12]

$$\frac{I_o'(x)}{x} = \frac{1}{2}[I_o(x) - I_2(x)], \tag{3.11}$$

one can prove that

$$R_\theta(x,t;\zeta,\theta) = \frac{\omega^2(\theta - t)}{2}\left[I_o\left(\omega\sqrt{(t-\theta)^2 - (\zeta - x)^2}\right) - I_2\left(\omega\sqrt{(t-\theta)^2 - (\zeta - x)^2}\right)\right], \tag{3.12}$$

where $I_2$ is the modified Bessel function of second order. Recalling that $U(x,0) = U_o(x)$ and, by $(3.3)_{(3)}$, (3.4), that

$$U_t(x,0) = \omega U_o(x), \tag{3.13}$$

we see that

$$\begin{aligned} &[R(x,t;\zeta,0)\omega - R_\theta(x,t;\zeta,0)]U_o(\zeta) \\ &\quad = \left[I_o\left(\omega\sqrt{t^2 - (\zeta - x)^2}\right)\left(\omega + \frac{\omega^2 t}{2}\right) - \frac{\omega^2 t}{2} I_2\left(\omega\sqrt{t^2 - (\zeta - x)^2}\right)\right]U_o(\zeta), \end{aligned} \tag{3.14}$$

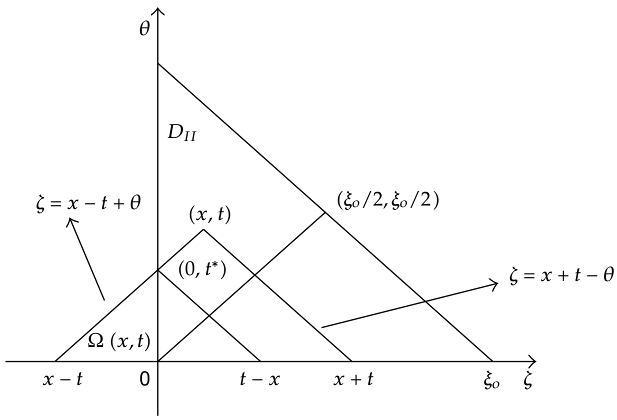


**Figure 4:** The domain $D_{II}$.

and representation formula (3.5) can be rewritten as

$$\begin{aligned} U(x,t) &= \frac{1}{2}[U_o(x-t)+U_o(x+t)] \\ &+ \frac{1}{2}\int_{x-t}^{x+t}\left[I_o\left(\omega\sqrt{t^2-(\zeta-x)^2}\right)\left(\omega+\frac{\omega^2 t}{2}\right)-\frac{\omega^2 t}{2}I_2\left(\omega\sqrt{t^2-(\zeta-x)^2}\right)\right]U_o(\zeta)d\zeta \\ &+ \frac{\beta^2}{2}\int_0^t \exp\left(\frac{k\theta}{2}\right)d\theta\cdot\int_{x-t+\theta}^{x+t-\theta} I_o\left(\omega\sqrt{(t-\theta)^2-(\zeta-x)^2}\right)d\zeta. \end{aligned} \tag{3.15}$$

Let us now write a representation formula for $U(x,t)$ in the domain $D_{II}$ (see Figure 4). We once again make use of (3.6), where now $U_o$ has to be extended to the domain $[-\xi_o, 0]$. Following [2], we extend $U_o$ imposing condition $(3.3)_{(4)}$, that is, $U(0,\theta) = 0$. From the representation formula we get

$$\begin{aligned} 0 &= \frac{1}{2}[U_o(-t^*)+U_o(t^*)] + \frac{1}{2}\int_{-t^*}^{t^*}[R(0,t^*;\zeta,0)\omega - R_\theta(0,t^*;\zeta,0)]U_o(\zeta)d\zeta \\ &+ \frac{\beta^2}{2}\int_0^{t^*}\exp(\omega\theta)d\theta\int_{-t^*+\theta}^{t^*-\theta}R(0,t^*;\zeta,\theta)d\zeta, \end{aligned} \tag{3.16}$$

where

$$t^* = t - x \tag{3.17}$$

is the coordinate of the intersection of the characteristic $\zeta = x - t + \theta$ with $\zeta = 0$. Relation (3.16) can be rewritten as

$$\begin{aligned} 0 &= \frac{1}{2}[U_o(x-t)+U_o(t-x)] + \frac{1}{2}\int_{x-t}^{t-x}[R(0,t-x;\zeta,0)\omega - R_\theta(0,t-x;\zeta,0)]U_o(\zeta)d\zeta \\ &+ \frac{\beta^2}{2}\int_0^{t-x}\exp(\omega\theta)d\theta\int_{x-t+\theta}^{t-x-\theta}R(0,t-x;\zeta,\theta)d\zeta. \end{aligned} \tag{3.18}$$

From (3.18), the extended function $U_o^{sx}(x)$, defined in $[-\xi_o, 0]$, fulfills the following Volterra integral equation of second type:

$$\begin{aligned}U_o^{sx}(x-t) &- \int_0^{x-t}[R(0,t-x;\zeta,0)\omega - R_\theta(0,t-x;\zeta,0)]U_o^{sx}(\zeta)d\zeta \\ &= -U_o(t-x) + \int_{t-x}^{0}[R(0,t-x;\zeta,0)\omega - R_\theta(0,t-x;\zeta,0)]U_o(\zeta)d\zeta \\ &\quad - \frac{\beta^2}{2}\int_0^{t-x}\exp(\omega\theta)d\theta\int_{x-t+\theta}^{t-x-\theta}R(0,t-x;\zeta,\theta)d\zeta.\end{aligned} \tag{3.19}$$

Equation (3.19) can be put in the more compact form

$$U_o^{sx}(\chi) - \int_0^{\chi}K^{sx}(\chi,\zeta)U_o^{sx}(\zeta)d\zeta = F^{sx}(\chi), \tag{3.20}$$

where $\chi = x - t \in [-\xi_o, 0]$ and

$$\begin{aligned}K^{sx}(\chi,\zeta) &= \left[I_o\left(\omega\sqrt{\chi^2-\zeta^2}\right)\left(\omega - \frac{\omega^2\chi}{2}\right) + \frac{\omega^2\chi}{2}I_2\left(\omega\sqrt{\chi^2-\zeta^2}\right)\right], \\ F^{sx}(\chi) &= -U_o(-\chi) + \int_{-\chi}^{0}[R(0,-\chi;\zeta,0)\omega - R_\theta(0,-\chi;\zeta,0)]U_o(\zeta)d\zeta \\ &\quad - \frac{\beta^2}{2}\int_0^{-x}\exp(\omega\theta)d\theta\int_{x+\theta}^{-x-\theta}R(0,-x;\zeta,\theta)d\zeta.\end{aligned} \tag{3.21}$$

Due to the regularity of the kernel $K^{sx}(\chi,\zeta)$ the function $U_o^{sx}(\chi)$ (which can be determined using the iterated kernels method, [13]) is a smooth function. Thus we extend $U_o(x)$ as

$$U_o(x) = \begin{cases} U_o^{sx}(x), & x \in [-\xi_o, 0], \\ U_o(x), & x \in [0, \xi_o], \end{cases} \tag{3.22}$$

and the solution $U(x,t)$ in the domain $D_{II}$ is given by

$$\begin{aligned}U(x,t) &= \frac{1}{2}[U_o(x+t) - U_o(t-x)] + \frac{1}{2}\int_{t-x}^{x-t}[R(0,t-x;\zeta,0)\omega - R_\theta(0,t-x;\zeta,0)]\,U_o(\zeta)d\zeta \\ &\quad + \frac{1}{2}\int_{x-t}^{x+t}[R(x,t;\zeta,0)\omega - R_\theta(t,x;\zeta,0)]U_o(\zeta)d\zeta + \frac{\beta^2}{2}\int_0^{t-x}e^{\omega\theta}d\theta\int_{t-x-\theta}^{x-t+\theta}R(0,t-x;\zeta,\theta)d\zeta \\ &\quad + \frac{\beta^2}{2}\int_0^{t}e^{\omega\theta}d\theta\int_{x-t+\theta}^{x+t-\theta}R(x,t;\zeta,\theta)d\zeta.\end{aligned} \tag{3.23}$$

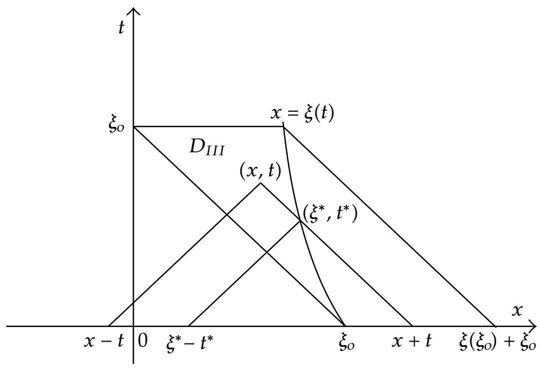


**Figure 5:** The domain $D_{III}$.

*Remark 3.1.* We notice that, considering the representation formulae (3.5) and (3.23),

$$\lim_{x\to t^+} U(x,t) = \lim_{x\to t^-} U(x,t). \tag{3.24}$$

Moreover, taking the first derivatives (with respect to time $t$ and space $x$) of $U(x,t)$ for the domains $D_I$ and $D_{II}$ it is easy to prove that, assuming the compatibility condition $U_o(0) = 0$,

$$\begin{aligned} \lim_{x\to t^+} U_x(x,t) &= \lim_{x\to t^-} U_x(x,t), \quad t \in \left[0, \frac{\xi_o}{2}\right], \\ \lim_{x\to t^+} U_t(x,t) &= \lim_{x\to t^-} U_t(x,t), \quad t \in \left[0, \frac{\xi_o}{2}\right], \end{aligned} \tag{3.25}$$

where the derivatives in limits on the l.h.s. of (3.25) are evaluated using (3.5), while the ones on the r.h.s. using (3.23). This implies that the solution is $C^1$ across the characteristic $x = t$, that is, the line that separates the domains $D_I$ and $D_{II}$.

We now write the representation formula for $U(x,t)$ in the domain $D_{III}$. We proceed as in [2] assuming that the velocity of the free boundary $x = \xi(t)$ is less than the velocity of the characteristics (i.e., $|\dot{\xi}| < 1$) and extending $U_o$ to the domain $[\xi_o, \xi(\xi_o) + \xi_o]$ (see Figure 2) in a way such that $U(\xi,t) = \exp(\omega t)V_o$ (i.e., imposing the free boundary condition $(3.3)_{(5)}$).

Given a point $(x,t)$ in the domain $D_{III}$ we define the point $(\xi^*, t^*)$ as the intersection of the characteristic (with negative slope) passing from $(x,t)$ and the free boundary $x = \xi(t)$ (see Figure 5). It is easy to check that

$$\xi^* + t^* = x + t, \quad \Longrightarrow t^* = t^*(x,t), \tag{3.26}$$

$$\frac{\partial t^*}{\partial t} = \frac{1}{\dot{\xi}(t^*) + 1}, \quad \frac{\partial t^*}{\partial x} = \frac{1}{\dot{\xi}(t^*) + 1}. \tag{3.27}$$

We consider once again the representation formula (3.5) and impose condition $(3.3)_{(5)}$, getting

$$\begin{aligned}2e^{\omega t^*}V_o &= U_o(\xi^*-t^*)+U_o(\xi^*+t^*)+\int_{\xi^*-t^*}^{\xi^*+t^*}[R(\xi^*,t^*;\zeta,0)\omega-R_\theta(\xi^*,t^*;\zeta,0)]U_o(\zeta)d\zeta\\&\quad+\beta^2\int_0^{t^*}e^{\omega\theta}d\theta\int_{\xi^*-t^*+\theta}^{\xi^*+t^*-\theta}R(\xi^*,t^*;\zeta,\theta)d\zeta.\end{aligned}\tag{3.28}$$

From (3.28) we see that the extension $U_o^{dx}$ to the domain $[\xi_o,\xi(\xi_o)+\xi_o]$ is the solution of the following Volterra integral equation of second kind:

$$\begin{aligned}&U_o^{dx}(\xi^*+t^*)+\int_{\xi_o}^{\xi^*+t^*}[R(\xi^*,t^*;\zeta,0)\omega-R_\theta(\xi^*,t^*;\zeta,0)]U_o^{dx}(\zeta)d\zeta\\&\quad=2e^{\omega t^*}V_o-U_o(\xi^*-t^*)-\int_{\xi^*-t^*}^{\xi_o}[R(\xi^*,t^*;\zeta,0)\omega-R_\theta(\xi^*,t^*;\zeta,0)]U_o(\zeta)d\zeta\\&\quad-\beta^2\int_0^{t^*}e^{\omega\theta}d\theta\int_{\xi^*-t^*+\theta}^{\xi^*+t^*-\theta}R(\xi^*,t^*;\zeta,\theta)d\zeta.\end{aligned}\tag{3.29}$$

Recalling (3.26) and proceeding as for the domain $D_{II}$, the above can be rewritten as

$$U_o^{dx}(\chi)+\int_{\xi_o}^{\chi}K^{dx}(\chi,\zeta)U_o^{dx}(\zeta)d\zeta=F^{dx}(\chi),\tag{3.30}$$

where $\chi=x+t$ and

$$\begin{aligned}K^{dx}(\chi,\zeta)&=\Bigg[I_o\left(\omega\sqrt{(t^*(\chi))^2-(\zeta-\xi^*(\chi))^2}\right)\left(\omega+\frac{\omega^2t^*(\chi)}{2}\right)\\&\qquad-\frac{\omega^2t^*(\chi)}{2}I_2\left(\omega\sqrt{(t^*(\chi))^2-(\xi^*(\chi)-\zeta)^2}\right)\Bigg],\\F^{dx}(\chi)&=\Bigg[2e^{\omega t}V_o-U_o(x-t)-\int_{x-t}^{\xi_o}[R(x,t;\zeta,0)\omega-R_\theta(x,t;\zeta,0)]U_o(\zeta)d\zeta\\&\qquad-\beta^2\int_0^t e^{\omega\theta}d\theta\int_{x-t+\theta}^{x+t-\theta}R(x,t;\zeta,\theta)d\zeta\Bigg]\Bigg|_{(x=\xi^*(\chi),t=t^*(\chi))}.\end{aligned}\tag{3.31}$$

Once again the regularity of the kernel $K^{dx}(\chi,\zeta)$ ensures the regularity of the solution $U_o^{dx}(\chi)$. The function $U_o(x)$ can thus be defined in the interval $[-\xi_o,\xi(\xi_o)+\xi_o]$ as

$$U_o(x)=\begin{cases}U_o^{sx}(x), & x\in[-\xi_o,0],\\U_o(x), & x\in[0,\xi_o],\\U_o^{dx}(x), & x\in[\xi_o,\xi(\xi_o)+\xi_o].\end{cases}\tag{3.32}$$

The solution in the domain $D_{III}$ is thus given by

$$\begin{aligned} U(x,t) = {} & e^{\omega t^*} V_o + \frac{1}{2}[U_o(x-t) - U_o(\xi^* - t^*)] + \frac{1}{2}\int_{x-t}^{x+t} P(x,t;\zeta,0)U_o(\zeta)d\zeta \\ & - \frac{1}{2}\int_{\xi^*-t^*}^{x+t} P(\xi^*,t^*;\zeta,0)U_o(\zeta)d\zeta + \frac{\beta^2}{2}\int_0^t e^{\omega\theta}d\theta \int_{x-t+\theta}^{x+t-\theta} R(x,t;\zeta,\theta)d\zeta \\ & - \frac{\beta^2}{2}\int_0^{t^*} e^{\omega\theta}d\theta \int_{\xi^*-t^*+\theta}^{x+t-\theta} R(\xi^*,t^*;\zeta,\theta)d\zeta, \end{aligned} \tag{3.33}$$

where for simplicity of notation we have introduced

$$P(x,t;\zeta,\theta) = R(x,t;\zeta,\theta)\omega - R_\theta(x,t;\zeta,\theta), \tag{3.34}$$

and where $U_o(x)$ is given by (3.32). Therefore for any fixed $C^1$ function $\xi(t)$ with $|\dot{\xi}| < 1$ we have that the solution to problem $(3.3)_{(1-5)}$ is given by (3.5), (3.23), (3.33) with $U_o$ defined by (3.32). At this point we make use of $(3.3)_{(6)}$ to determine the evolution equation of the free boundary $x = \xi(t)$. We begin writing the derivatives $U_t(x,t)$ and $U_x(x,t)$. To this aim we exploit formula (3.33) since $U_t$ and $U_x$ have to be evaluated on $x = \xi(t)$, which belongs to domain $D_{III}$. Differentiating (3.33) with respect to $x$ we get

$$\begin{aligned} U_x(x,t) = {} & \frac{1}{2}\left[U_o'(x-t) - U_o'(\xi^*-t^*)\frac{\dot{\xi}(t^*)-1}{\dot{\xi}(t^*)+1}\right] + \left(\frac{\omega V_o e^{\omega t^*}}{\dot{\xi}^*+1}\right) \\ & + \frac{1}{2}[P(x,t;x+t,0)U_o(x+t) - P(x,t;x-t,0)U_o(x-t)] \\ & - \frac{1}{2}\left[P(\xi^*,t^*;x+t,0)U_o(x+t) - P(\xi^*,t^*;\xi^*-t^*,0)U_o(\xi^*-t^*)\frac{\dot{\xi}(t^*)-1}{\dot{\xi}(t^*)+1}\right] \\ & - \frac{1}{2}\int_{\xi^*-t^*}^{x+t}[P_x(\xi^*,t^*;\zeta,0)\dot{\xi}(t^*) + P_t(\xi^*,t^*;\zeta,0)]\frac{U_o(\zeta)}{\dot{\xi}(t^*)+1}d\zeta \\ & + \frac{1}{2}\int_{x-t}^{x+t} P_x(x,t;\zeta,0)U_o(\zeta)d\zeta + \frac{\beta^2}{2}\int_0^t e^{\omega\theta}d\theta\int_{x-t+\theta}^{x+t-\theta} R_x(x,t;\zeta\theta)d\zeta \\ & - \frac{\beta^2}{2}\int_0^{t^*} e^{\omega\theta}\left[R(\xi^*,t^*;x+t-\theta,\theta) - R(\xi^*,t^*;\xi^*-t^*+\theta,\theta)\frac{\dot{\xi}(t^*)-1}{\dot{\xi}(t^*)+1}\right]d\theta \\ & - \frac{\beta^2}{2}\int_0^{t^*} e^{\omega\theta}\int_{\xi^*-t^*+\theta}^{x+t-\theta}[R_x(\xi^*,t^*;\zeta,\theta)\dot{\xi}(t^*) - R_t(\xi^*,t^*;\zeta,\theta)]\frac{d\zeta}{\dot{\xi}(t^*)+1}, \end{aligned} \tag{3.35}$$

while, differentiating (3.33) with respect to $t$, we obtain

$$\begin{aligned}
U_t(x,t) &= \frac{1}{2}\left[-U_o'(x-t) - U_o'(\xi^*-t^*)\frac{\dot{\xi}(t^*)-1}{\dot{\xi}(t^*)+1}\right] + \left(\frac{\omega V_o e^{\omega t^*}}{\xi^*+1}\right) \\
&+ \frac{1}{2}[P(x,t;x+t,0)U_o(x+t) + P(x,t;x-t,0)U_o(x-t)] \\
&- \frac{1}{2}\left[P(\xi^*,t^*;x+t,0)U_o(x+t) - P(\xi^*,t^*;\xi^*-t^*,0)U_o(\xi^*-t^*)\frac{\dot{\xi}(t^*)-1}{\dot{\xi}(t^*)+1}\right] \\
&- \frac{1}{2}\int_{\xi^*-t^*}^{x+t}[P_x(\xi^*,t^*;\zeta,0)\dot{\xi}(t^*) + P_t(\xi^*,t^*;\zeta,0)]\frac{U_o(\zeta)}{\dot{\xi}(t^*)+1}d\zeta + \beta^2\int_0^t e^{\omega\theta}d\theta \\
&+ \frac{1}{2}\int_{x-t}^{x+t}P_t(x,t;\zeta,0)U_o(\zeta)d\zeta + \frac{\beta^2}{2}\int_0^t e^{\omega\theta}d\theta\int_{x-t+\theta}^{x+t-\theta}R_t(x,t;\zeta\theta)d\zeta \\
&- \frac{\beta^2}{2}\int_0^{t^*}e^{\omega\theta}\left[R(\xi^*,t^*;x+t-\theta,\theta) - R(\xi^*,t^*;\xi^*-t^*+\theta,\theta)\frac{\dot{\xi}(t^*)-1}{\dot{\xi}(t^*)+1}\right]d\theta \\
&- \frac{\beta^2}{2}\int_0^{t^*}e^{\omega\theta}\int_{\xi^*-t^*+\theta}^{x+t-\theta}[R_x(\xi^*,t^*;\zeta,\theta)\dot{\xi}(t^*) - R_t(\xi^*,t^*;\zeta,\theta)]\frac{d\zeta}{\dot{\xi}(t^*)+1}.
\end{aligned} \tag{3.36}$$

Notice that

$$\begin{aligned}
P(x,t;x+t,0) &= P(x,t;x-t,0) = \omega + \frac{\omega^2 t}{2}, \\
R(x,t;x+t-\theta,\theta) &- R(x,t;x-t+\theta,\theta) = 0.
\end{aligned} \tag{3.37}$$

Now we evaluate (3.35) and (3.36) on the free boundary $x = \xi(t)$, that is,

$$\begin{aligned}
U_x(\xi,t) &= \frac{U_o'(\xi-t)}{\dot{\xi}+1} - \left(\omega + \frac{\omega^2 t}{2}\right)U_o(\xi-t)\frac{1}{\dot{\xi}+1} + \left(\frac{\omega V_o e^{\omega t}}{\dot{\xi}+1}\right) \\
&+ \frac{1}{2}\int_{\xi-t}^{\xi+t}[P_x(\xi,t;\zeta,0) - P_t(\xi,t;\zeta,0)]\frac{U_o(\zeta)d\zeta}{\dot{\xi}+1} - \beta^2\int_0^t\frac{e^{\omega\theta}}{\dot{\xi}+1}d\theta \\
&+ \frac{\beta^2}{2}\int_0^t e^{\omega\theta}\int_{\xi-t+\theta}^{\xi+t-\theta}[R_x(\xi,t;\zeta,\theta) - R_t(\xi,t;\zeta,\theta)]\frac{d\zeta}{\dot{\xi}+1}, \\
U_t(\xi,t) &= -\frac{U_o'(\xi-t)\dot{\xi}}{\dot{\xi}+1} + \left(\omega + \frac{\omega^2 t}{2}\right)U_o(\xi-t)\frac{\dot{\xi}}{\dot{\xi}+1} + \left(\frac{\omega V_o e^{\omega t}}{\dot{\xi}+1}\right) \\
&+ \frac{1}{2}\int_{\xi-t}^{\xi+t}[P_t(\xi,t;\zeta,0) - P_x(\xi,t;\zeta,0)]\frac{\dot{\xi}U_o(\zeta)d\zeta}{\dot{\xi}+1} + \beta^2\int_0^t\frac{e^{\omega\theta}\dot{\xi}}{\dot{\xi}+1}d\theta \\
&+ \frac{\beta^2}{2}\int_0^t e^{\omega\theta}\int_{\xi-t+\theta}^{\xi+t-\theta}[R_t(\xi,t;\zeta,\theta) - R_x(\xi,t;\zeta,\theta)]\frac{\dot{\xi}d\zeta}{\dot{\xi}+1}.
\end{aligned} \tag{3.38}$$

At this point we insert (3.38), $(3.3)_{(5)}$ in $(3.3)_{(6)}$, obtaining

$$
\begin{aligned}
(\dot{\xi}-1)\Bigg[\left(\omega+\frac{\omega^2 t}{2}\right)U_o(\xi-t)-U_o'(\xi-t)+\frac{\beta^2}{\omega}(e^{\omega t}-1)+\frac{1}{2}\int_{\xi-t}^{\xi+t}[P_t(\xi,t;\zeta,0)-P_x(\xi,t;\zeta,0)]U_o(\zeta)d\zeta \\
+\frac{\beta^2}{2}\int_0^t e^{\omega\theta}d\theta\int_{\xi-t+\theta}^{\xi+t-\theta}[R_t(\xi,t;\zeta,\theta)-R_x(\xi,t;\zeta,\theta)]d\zeta\Bigg] = e^{\omega t}\beta^2\mathrm{Bn},
\end{aligned}
\tag{3.39}
$$

which is a nonlinear integrodifferential equation of the first order and where $U_o$ is defined by (3.32). Equation (3.39) is the free boundary equation which, as we mentioned in the introduction, does no longer depend on the velocity field $U(x,t)$.

Next we remark that (3.39) can be further simplified. Indeed, recalling (3.10) and (3.12),

$$
R_x(x,t;\zeta,\theta) = -R_\zeta(x,t;\zeta,\theta), \quad R_{\theta x}(x,t;\zeta,\theta) = -R_{\theta\zeta}(x,t;\zeta,\theta), \tag{3.40}
$$

so that, on $(\xi(t),t;\zeta,\theta)$, we have

$$
\int_{\xi-t+\theta}^{\xi+t-\theta} R_x d\zeta = -\int_{\xi-t+\theta}^{\xi+t-\theta} R_\zeta d\zeta = R(\xi,t;\xi+t-\theta,\theta) - R(\xi,t;\xi-t+\theta,\theta) = 0, \tag{3.41}
$$

while, on $(\xi(t),t;\zeta,\theta)$,

$$
\begin{aligned}
\int_{\xi-t}^{\xi+t} P_x d\zeta &= -\int_{\xi-t}^{\xi+t} R_\zeta \omega d\zeta + \int_{\xi-t}^{\xi+t} R_{\theta\zeta} d\zeta \\
&= \omega[R(\xi,t;\xi-t,0) - R(\xi,t;\xi+t,0)] + [R_\theta(\xi,t;\xi+t,0) - R_\theta(\xi,t;\xi-t,0)] = 0.
\end{aligned}
\tag{3.42}
$$

Hence (3.39) reduces to

$$
\begin{aligned}
(\dot{\xi}-1)\Bigg[\left(\omega+\frac{\omega^2 t}{2}\right)U_o(\xi-t) - U_o'(\xi-t) + \frac{\beta^2}{\omega}(e^{\omega t}-1) \\
+\frac{1}{2}\int_{\xi-t}^{\xi+t} P_t(\xi,t;\zeta,0)U_o(\zeta)d\zeta + \frac{\beta^2}{2}\int_0^t e^{\omega\theta}d\theta\int_{\xi-t+\theta}^{\xi+t-\theta} R_t(\xi,t;\zeta,\theta)d\zeta\Bigg] = e^{\omega t}\beta^2\mathrm{Bn}.
\end{aligned}
\tag{3.43}
$$

*Remark 3.2.* The function $U(x,t)$ is continuous across the characteristic $x+t=\xi_o$. Indeed

$$\lim_{x+t\to\xi_o^+} U(x,t) = \lim_{x+t\to\xi_o^-} U(x,t), \tag{3.44}$$

where the limit $\lim_{x+t\to\xi_o^+}$ is evaluated using (3.33) and the limit $\lim_{x+t\to\xi_o^-}$ using (3.5) or (3.23). If we evaluate the derivatives $U_x$ and $U_t$ on the characteristic $x+t=\xi_o$ we get two different results depending on whether we are evaluating such derivatives in $D_I$ or $D_{III}$. We can prove that

$$\begin{aligned}\lim_{x+t\to\xi_o^-} U_x(x,t) &= \frac{1}{2}\left[U_o'(x-t)+U_o'(\xi_o)+P(x,t;\xi_o,0)U_o(\xi_o)-P(x,t;x-t,0)U_o(x-t)\right]\\ &\quad+\frac{1}{2}\int_{x-t}^{\xi_o} P_x(x,t;\zeta,0)U_o(\zeta)d\zeta+\frac{\beta^2}{2}\int_0^t e^{\omega\theta}d\theta\int_{x-t+\theta}^{\xi_o-\theta} R_x(x,t;\zeta,\theta)d\zeta,\end{aligned} \tag{3.45}$$

$$\begin{aligned}\lim_{x+t\to\xi_o^+} U_x(x,t) &= \frac{1}{2}\left[U_o'(x-t)-U_o'(\xi_o)\frac{\dot{\xi}_o-1}{\dot{\xi}_o+1}+P(x,t;\xi_o,0)U_o(\xi_o)-P(x,t;x-t,0)U_o(x-t)\right]\\ &\quad+\frac{\omega V_o}{\dot{\xi}_o+1}+\frac{1}{2}\int_{x-t}^{\xi_o} P_x(x,t;\zeta,0)U_o(\zeta)d\zeta+\frac{\beta^2}{2}\int_0^t e^{\omega\theta}d\theta\int_{x-t+\theta}^{\xi_o-\theta} R_x(x,t;\zeta,\theta)d\zeta,\end{aligned} \tag{3.46}$$

$$\begin{aligned}\lim_{x+t\to\xi_o^-} U_t(x,t) &= \frac{1}{2}\left[U_o'(\xi_o)-U_o'(x-t)+P(x,t;\xi_o,0)U_o(\xi_o)+P(x,t;x-t,0)U_o(x-t)\right]\\ &\quad+\beta^2\int_0^t e^{\omega\theta}d\theta+\frac{1}{2}\int_{x-t}^{\xi_o} P_t(x,t;\zeta,0)U_o(\zeta)d\zeta+\frac{\beta^2}{2}\int_0^t e^{\omega\theta}d\theta\int_{x-t+\theta}^{\xi_o-\theta} R_t(x,t;\zeta,\theta)d\zeta,\end{aligned} \tag{3.47}$$

$$\begin{aligned}\lim_{x+t\to\xi_o^+} U_t(x,t) &= \frac{1}{2}\left[-U_o'(x-t)-U_o'(\xi_o)\frac{\dot{\xi}_o-1}{\dot{\xi}_o+1}+P(x,t;\xi_o,0)U_o(\xi_o)-P(x,t;x-t,0)U_o(x-t)\right]\\ &\quad+\frac{\omega V_o}{\dot{\xi}_o+1}+\frac{1}{2}\int_{x-t}^{\xi_o} P_t(x,t;\zeta,0)U_o(\zeta)d\zeta+\beta^2\int_0^t e^{\omega\theta}d\theta\\ &\quad+\frac{\beta^2}{2}\int_0^t e^{\omega\theta}d\theta\int_{x-t+\theta}^{x+t-\theta} R_x(x,t;\zeta,\theta)d\zeta,\end{aligned} \tag{3.48}$$

where $\dot{\xi}_o=\dot{\xi}(0)$. It is easy to check that, imposing that (3.45) equals (3.46) and that (3.47) equals (3.48), we get the following condition:

$$\dot{\xi}_o U_o'(\xi_o) = \omega U_o(\xi_o) = \omega V_o, \tag{3.49}$$

which is the condition that must be fulfilled if we want the first derivatives of $U(x,t)$ to be continuous across the characteristic $x + t = \xi_o$.

If we assume that the free boundary equation $(3.3)_{(6)}$ holds up to $t = 0$ we get

$$U'_o(\xi_o) = \beta^2 \mathtt{Bn}. \tag{3.50}$$

Moreover, from (3.43) we have that, when $t = 0$,

$$(\dot{\xi}_o - 1)\left[\omega U_o(\xi_o) - U'_o(\xi_o)\right] = \beta^2 \mathtt{Bn}. \tag{3.51}$$

We can therefore prove the following.

**Theorem 3.3.** *If one assumes that compatibility condition $U_o(\xi_o) = V_o$ and hypotheses* (3.49), (3.50), (3.51) *hold, then necessarily either $V_o = 0$ or $\omega = 0$ and problem* (3.3) *admits a unique local $C^1$ solution $(U, \xi)$, such that $\dot{\xi}(0) = 0$. If one does not assume hypothesis* (3.49) *(meaning that the first derivatives of $U$ are not continuous along the characteristic $x + t = \xi_o$), then problem* (3.3) *admits a unique local solution $(U, \xi)$, such that $-1 < \dot{\xi}(t) < 0$ if and only if*

$$V_o < \frac{\beta^2 \mathtt{Bn}}{2\omega}. \tag{3.52}$$

*Proof.* If we suppose that (3.49), (3.50), (3.51) hold then we have

$$(\dot{\xi}_o - 1)\left[U'_o(\xi_o)\dot{\xi}_o - U'_o(\xi_o)\right] = \beta^2 \mathtt{Bn}, \quad \Longrightarrow \dot{\xi}_o = 0 \ \text{ or } \dot{\xi}_o = 2. \tag{3.53}$$

The initial velocity $\dot{\xi}_o = 2$ is not physically acceptable since existence of a solution requires that $|\dot{\xi}| < 1$. Therefore $\dot{\xi}_o = 0$ and, recalling (3.49), we have either $V_o = 0$ or $\omega = 0$, since $U'_o(\xi_o) = \beta^2 \mathtt{Bn} \neq \pm\infty$. If, on the other hand, we suppose that condition (3.49) does not hold, but we assume (3.52), then it is easy to show that

$$-1 < \frac{\omega V_o}{\omega V_o - \beta^2 \mathtt{Bn}} = 1 + \frac{\beta^2 \mathtt{Bn}}{\omega V_o - \beta^2 \mathtt{Bn}} = \dot{\xi}_o < 0, \tag{3.54}$$

so that for a sufficiently small time $t > 0$ there exists a unique solution with $-1 < \dot{\xi} < 0$. The existence of such a solution can be proved using classical tools like iterated kernels method (see [13]). □

*Remark 3.4.* Let us consider the limit case in which $\omega = 0$ and $\beta^2 = 0$. In this particular situation the Riemann's function $R(x,t;\zeta,\theta) \equiv 1$ and the solution $U(x,t)$ is given by

$$U(x,t) = \begin{cases} \dfrac{1}{2}[U_o(x+t) + U_o(x-t)], & \text{in } D_I, \\ \dfrac{1}{2}[U_o(x+t) - U_o(t-x)], & \text{in } D_{II}, \\ \dfrac{1}{2}[U_o(x-t) - U_o(\xi-t)] + V_o, & \text{in } D_{III}, \end{cases} \tag{3.55}$$

and the free boundary equation is the characteristic with positive slope passing through $(\xi_o, 0)$, that is,

$$(\dot{\xi} - 1)U_o'(\xi - t) = 0 \Longrightarrow U_o(\xi - t) = U_o(\xi_o), \tag{3.56}$$

namely,

$$\xi(t) = \xi_o + t \Longrightarrow \dot{\xi}(t) = 1. \tag{3.57}$$

So, setting $t_o = 1 - \xi_o$, for $t \geq t_o$, the region with uniform velocity (the inner core) has disappeared. For $t \geq t_o$, the solution $U(x,t)$ is thus found solving

$$\begin{gathered} U_{xx} = U_{tt}, \quad 0 \leqslant x \leqslant 1, \;\; t \geqslant 1 - \xi_o \\ U(x, 1-\xi_o) = U_o^*(x), \quad 0 \leqslant x \leqslant 1, \\ U_t(x, 1-\xi_o) = U_1^*(x), \quad 0 \leqslant x \leqslant 1, \\ U(0,t) = 0 \quad t \geqslant 1 - \xi_o, \\ U(1,t) = V_o \quad t \geqslant 1 - \xi_o, \end{gathered} \tag{3.58}$$

where $U_o^*(x)$ and $U_1^*(x)$ are determined evaluating (3.55) at time $t = t_o$. To solve problem (3.58) we introduce the new variable

$$W(x,t) = U(x,t) - xV_o \tag{3.59}$$

and rescale time with

$$\theta = t - t_o. \tag{3.60}$$

Problem (3.58) becomes

$$\begin{gathered} W_{xx} = W_{\theta\theta}, \quad 0 \leqslant x \leqslant 1, \; \theta \geqslant 0, \\ W(x,0) = U_o^*(x) - V_o(x), \quad 0 \leqslant x \leqslant 1, \\ W_\theta(x,0) = W_1(x) = U_1^*(x), \quad 0 \leqslant x \leqslant 1, \\ W(0,\theta) = 0, \quad \theta \geqslant 0, \\ W(1,\theta) = 0, \quad \theta \geqslant 0, \end{gathered} \tag{3.61}$$

whose solution is [11]

$$W(x,\theta) = \sum_{i=1}^{\infty}[A_n\cos(\pi n\theta) + B_n\sin(\pi n\theta)]\sin(\pi nx), \tag{3.62}$$

where

$$A_n = 2\int_0^{\theta} W_o(z)\sin(\pi nz)dz, \qquad B_n = \frac{2}{\pi n}\int_0^{\theta} W_1(z)\sin(\pi nz)dz,$$
$$U(x,t) = W(x,t-1+\xi_o) + xV_o. \tag{3.63}$$

## 4. Asymptotic Expansion

In this section we look for a solution to problem $(3.3)_{(1)}$ in the following form:

$$U(x,t) = \sum_{i=0}^{\infty}\omega^i U^{(i)}(x,t). \tag{4.1}$$

This allows to obtain a sequence of problems for each $i = 0,1,2\ldots$ with the free boundary being given by $\xi^{(i)}(t)$. (We remark that the sequence $\{\xi^{(i)}(t)\}$ is not, in general, an asymptotic sequence.) Such an analysis is motivated by the fact that, in practical cases (asphalt and bitumen), $\omega = O(10^{-1})$ (see (2.3)). Hence, it makes sense to look for a "perturbative" approach for the system (3.3).

We do not discuss the issue of the convergence of series (4.1) and of the sequence $\{\xi^{(i)}(t)\}$, which is beyond the scope of the present paper. We limit ourselves to a formal derivation of the free boundary problems that can be obtained plugging $(4.1)_{(1)}$ into (3.3):

$$\sum_{i=0}^{\infty}\left[\omega^i U_{xx}^{(i)}(x,t) - \omega^i U_{tt}^{(i)}(x,t) + \omega^{i+2}U^{(i)}(x,t)\right] = -\beta^2\sum_{i=0}^{\infty}\frac{(\omega t)^i}{i!}. \tag{4.2}$$

Hence, for each $i = 0,1,2,\ldots$, we have

$$\begin{aligned}
i = 0,\quad & U_{xx}^{(o)}(x,t) - U_{tt}^{(o)}(x,t) = -\beta^2,\\
i = 1,\quad & U_{xx}^{(1)}(x,t) - U_{tt}^{(1)}(x,t) = -\beta^2 t,\\
i = 2,\quad & U_{xx}^{(2)}(x,t) - U_{tt}^{(2)}(x,t) = -\beta^2\frac{t^2}{2!} - U^{(o)}(x,t),\\
& \vdots\\
i > 2,\quad & U_{xx}^{(i)}(x,t) - U_{tt}^{(i)}(x,t) = -\beta^2\frac{t^i}{i!} - U^{(i-2)}(x,t)
\end{aligned} \tag{4.3}$$

and the following free boundary problems

$$i=0,\quad \begin{cases} U_{xx}^{(o)}(x,t)-U_{tt}^{(o)}(x,t)=-\beta^2 \\ U^{(o)}(x,0)=U_o(x), \\ U_t^{(o)}(x,0)=0, \\ U^{(o)}(0,t)=0, \\ U^{(o)}\left(\xi^{(o)},t\right)=V_o \\ U_x^{(o)}\left(\xi^{(o)},t\right)+\dot{\xi}^{(o)}U_t^{(o)}\left(\xi^{(o)},t\right)=\beta^2\mathtt{Bn}, \\ \xi^{(o)}(0)=\xi_o, \end{cases} \tag{4.4}$$

$$i=1,\quad \begin{cases} U_{xx}^{(1)}(x,t)-U_{tt}^{(1)}(x,t)=-t\beta^2 \\ U^{(1)}(x,0)=0, \\ U_t^{(1)}(x,0)=U_o(x), \\ U^{(1)}(0,t)=0, \\ U^{(1)}\left(\xi^{(1)},t\right)=V_o t \\ U_x^{(1)}\left(\xi^{(1)},t\right)+\dot{\xi}^{(1)}U_t^{(1)}\left(\xi^{(1)},t\right)-\dot{\xi}^{(1)}V_o=t\beta^2\mathtt{Bn}, \\ \xi^{(1)}(0)=\xi_o, \end{cases} \tag{4.5}$$

$$i\geq 2,\quad \begin{cases} U_{xx}^{(i)}(x,t)-U_{tt}^{(i)}(x,t)=-\dfrac{t^i}{i!}\beta^2-U^{(i-2)}(x,t) \\ U^{(i)}(x,0)=0, \\ U_t^{(i)}(x,0)=0, \\ U^{(i)}(0,t)=0, \\ U^{(i)}\left(\xi^{(i)},t\right)=\dfrac{t^i}{i!}V_o \\ U_x^{(i)}\left(\xi^{(i)},t\right)+\dot{\xi}^{(i)}U_t^{(i)}\left(\xi^{(i)},t\right)-\dot{\xi}^{(i)}V_o\dfrac{t^{i-1}}{(i-1)!}=\dfrac{t^i}{i!}\beta^2\mathtt{Bn}, \\ \xi^{(i)}(0)=\xi_o. \end{cases} \tag{4.6}$$

We immediately remark that, in each problem, the governing equation is no longer a telegrapher's equation, but a nonhomogeneous wave equation. Hence, using classical d'Alembert formula, we can write the representation formula for each domain $D_I$, $D_{II}$, $D_{III}$ and for each order of approximation $i = 0, 1, \ldots$. In particular, in $D_I$ we have

$$D_I \begin{cases} U^{(o)}(x,t) = \dfrac{1}{2}\left[U_o(x+t) + U_o(x-t)\right] + \dfrac{\beta^2 t^2}{2!}, \\ U^{(1)}(x,t) = \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} U_o(\zeta)d\zeta + \dfrac{\beta^2 t^3}{3!}, \\ \vdots \\ U^{(i)}(x,t) = \dfrac{\beta^2}{2}\displaystyle\int_0^t \int_{x-t+\theta}^{x+t-\theta} U^{(i-2)}(\zeta,\theta)d\zeta d\theta + \dfrac{\beta^2\ t^{i+2}}{(i+2)!}, \quad i \geqslant 2, \end{cases} \tag{4.7}$$

while in $D_{II}$

$$D_{II} \begin{cases} U^{(o)}(x,t) = \dfrac{1}{2}\left[U_o(x+t) - U_o(t-x)\right] + \dfrac{\beta^2 t^2}{2!} - \dfrac{\beta^2 (x-t)^2}{2!}, \\ U^{(1)}(x,t) = \dfrac{1}{2}\displaystyle\int_{t-x}^{x+t} U_o(\zeta)d\zeta + \dfrac{\beta^2 t^3}{3!} - \dfrac{\beta^2 (t-x)^3}{3!}, \\ \vdots \\ U^{(i)}(x,t) = \dfrac{\beta^2}{2}\displaystyle\int_0^t \int_{t-x+\theta}^{x-t+\theta} U^{(i-2)}(\zeta,\theta)d\zeta d\theta + \dfrac{\beta^2}{2}\int_0^t \int_{x-t+\theta}^{x+t-\theta} U^{(i-2)}(\zeta,\theta)d\zeta d\theta \\ \qquad + \dfrac{\beta^2 t^{i+2}}{(i+2)!} - \dfrac{\beta^2 (t-x)^{i+2}}{(i+2)!}, \quad i \geqslant 2, \end{cases} \tag{4.8}$$

and in $D_{III}$

$$D_{III} \begin{cases} U^{(o)}(x,t) = V_o + \dfrac{1}{2}\left[U_o(x-t) - U_o\left(\xi^{(o)*} - t^*\right)\right] + \dfrac{\beta^2 t^2}{2!} - \dfrac{\beta^2 t^{*2}}{2!}, \\ U^{(1)}(x,t) = V_o t^* + \dfrac{1}{2}\displaystyle\int_{x-t}^{\xi^{(1)*}-t^*} U_o(\zeta)d\zeta + \dfrac{\beta^2 t^3}{3!} - \dfrac{\beta^2 t^{*3}}{3!}, \\ \vdots \\ U^{(i)}(x,t) = \dfrac{\beta^2}{2}\displaystyle\int_0^t \int_{x-t+\theta}^{x+t-\theta} U^{(j-2)}(\zeta,\theta)d\zeta d\theta - \dfrac{\beta^2}{2}\int_0^{t^*} \int_{\xi^{(i)*}-t^*+\theta}^{x+t-\theta} U^{(i-2)}(\zeta,\theta)d\zeta d\theta \\ \qquad + V_o \dfrac{(t^*)i}{i!} + \dfrac{\beta^2 t^{i+2}}{(i+2)!} - \dfrac{\beta^2 (t^*)^{i+2}}{(i+2)!}. \quad i \geq 2. \end{cases} \tag{4.9}$$

Proceeding as in Section 3 we can show that the evolution equations of the free boundary $x = \xi^{(i)}(t)$ at each step are given by

$$i = 0, \qquad \left(\dot{\xi}^{(0)} - 1\right)\left[\beta^2 t - U_o'\left(\xi^{(o)} - t\right)\right] = \beta^2 \mathtt{Bn}, \tag{4.10}$$

$$i = 1, \qquad \left(\dot{\xi}^{(1)} - 1\right)\left[U_o\left(\xi^{(1)} - t\right) - V_o + \beta^2 \frac{t^2}{2!}\right] = t\beta^2 \mathtt{Bn}, \tag{4.11}$$

$$i \geqslant 2, \qquad \left(\dot{\xi}^{(i)} - 1\right)\left[\frac{\beta^2 (i+2) t^{i+1}}{(i+1)!} - \frac{V_o t^{i-1}}{(i-1)!} + \beta^2 \int_0^t U^{(i-2)}\left(\xi^{(i)} - t + \theta, \theta\right) d\theta\right] = \frac{t^i}{i!}\beta^2 \mathtt{Bn}. \tag{4.12}$$

At the zero order, assuming the compatibility condition $U'(\xi_o) = \beta^2 \mathrm{Bn}$ (see problem (4.4)), we have

$$\left(1 - \dot{\xi}_o^{(o)}\right) U_o'(\xi_o) = \beta^2 \mathtt{Bn}, \quad \Longrightarrow \dot{\xi}_o^{(o)} = 0. \tag{4.13}$$

At the first order (see problem (4.5)), we assume that the compatibility condition of second order holds in the corner $(\xi_o, 0)$. This means that we can differentiate the free boundary equation $(4.5)_{(6)}$ and take the limit for $t \to 0$. We have

$$U_{xx}^{(1)}\left(\xi^{(1)}, t\right)\dot{\xi}^{(1)} + U_{xt}^{(1)}\left(\xi^{(1)}, t\right) + \ddot{\xi}^{(1)} U_t^{(1)}\left(\xi^{(1)}, t\right) + \dot{\xi}^{(1)^2} U_{xt}^{(1)}\left(\xi^{(1)}, t\right) - \ddot{\xi}^{(1)} V_o = \beta^2 \mathtt{Bn}, \tag{4.14}$$

which, when $t \to 0$, reduces to

$$U_o^{'}(\xi_o)\left(1 + \dot{\xi}_o^{(1)^2}\right) = \beta^2 \mathtt{Bn}, \quad \Longrightarrow \dot{\xi}_o^{(1)} = 0. \tag{4.15}$$

For the generic $i$th order (see problem (4.6)), we assume that the compatibility conditions in the corner $(\xi_o, 0)$ hold up to order $(i-1)$. Therefore we can take the $(i-1)$th derivative of $(4.6)_{(6)}$, obtaining

$$\begin{aligned} &\frac{d^{i-1}}{dt^{i-1}}\left[U_x^{(i)}\left(\xi^{(i)}, t\right)\right] + \frac{d^{i-1}}{dt^{i-1}}\left[\dot{\xi}^{(i)}\right] U_t^{(i)}\left(\xi^{(i)}, t\right) + \dot{\xi}^{(i)} \frac{d^{i-1}}{dt^{i-1}}\left[U_t^{(i)}\left(\xi^{(i)}, t\right)\right] \\ &\quad - \dot{\xi}^{(i)} V_o - V_o \frac{t^{i-1}}{(i-1)!} \frac{d^{i-1}}{dt^{i-1}}\left[\dot{\xi}^{(i)}\right] = t\beta^2 \mathtt{Bn}, \end{aligned} \tag{4.16}$$

which, in the limit $t \to 0$, reduces to

$$-\dot{\xi}_o^{(i)} V_o = 0 \Longrightarrow \dot{\xi}_o^{(i)} = 0. \tag{4.17}$$

We therefore conclude that, assuming enough regularity for each problem $i = 0, 1, 2, \ldots$, (4.10)–(4.12) posses a unique local solution with $\xi^{(i)}(0) = \xi_o$ and $\dot{\xi}^{(i)}(0) = 0$.

Before proceeding further we suppose that $U(x)$ has the following properties:

(H1) $U_o(x) \in C^\infty([0,\xi_o])$,

(H2) $0 < U_o(x) < V_o$ for all $x \in (0,\xi_o)$, $U_o(0) = 0$, $U_o(\xi_o) = V_o$,

(H3) $U_o'(x) > 0$ for all $x \in [0,\xi_o]$ and $U'(\xi_o) = \beta^2\mathtt{Bn}$,

(H4) $U_o(x)$ satisfies all the compatibility conditions up to any order in the corner $(\xi_o, 0)$.

### 4.1. Zero-Order Approximation

We introduce the new variable $\phi^{(o)} = \xi^{(o)} - t$, so that (4.10) can be rewritten as

$$\dot{\phi}^{(o)}\left[\beta^2 t - U_o'\left(\phi^{(o)}\right)\right] = \beta^2\mathtt{Bn}, \tag{4.18}$$

with $\phi^{(o)}(0) = \xi_o$. Then we look for the solution $t = t(\phi^{(o)})$ which fulfills the following Cauchy problem:

$$\begin{aligned} &\beta^2\mathtt{Bn}\frac{dt}{d\phi^{(o)}} = \left[\beta^2 t - U_o'\left(\phi^{(o)}\right)\right], \\ &t(\xi_o) = 0, \end{aligned} \tag{4.19}$$

that is,

$$t\left(\phi^{(o)}\right) = -\frac{1}{\beta^2\mathtt{Bn}}\int_{\xi_o}^{\phi^{(o)}} U_o'(z)\exp\left(\frac{\phi^{(o)} - z}{\mathtt{Bn}}\right)dz. \tag{4.20}$$

Recalling that $|\dot{\xi}^{(o)}| < 1$, that is,

$$\left|\dot{\phi}^{(o)} + 1\right| < 1, \quad \Longrightarrow -2 < \dot{\phi}^{(o)} < 0, \quad \Longrightarrow \frac{dt}{d\phi^{(o)}} < -\frac{1}{2}, \tag{4.21}$$

from $(4.19)_{(1)}$ we realize that (4.21) is fulfilled if

$$\xi_o + \frac{\mathtt{Bn}}{2} < \frac{1}{\beta^2}\inf_{z\in[0,\xi_o]} U_o'(z). \tag{4.22}$$

Therefore, under hypothesis (4.22), local existence of a classical solution is guaranteed. Such a solution is given by $\xi^{(o)}(t) = \phi^{(o)}(t) + t$, where $\phi^{(o)}(t)$ is determined inverting (4.20).

### *4.2. First-Order Approximation*

We now have to solve the problem

$$\left(\dot{\xi}^{(1)}-1\right)\left[U_o\left(\xi^{(1)}-t\right)-V_o+\beta^2\frac{t^2}{2!}\right]=t\beta^2\mathtt{Bn}, \tag{4.23}$$

with $\xi^{(1)}(0)=\xi_o$. Proceeding as in Section 4.1 we introduce the new variable $\phi^{(1)}=\xi^{(1)}-t$, so that (4.23) becomes

$$\dot{\phi}^{(1)}\left[U_o\left(\phi^{(1)}\right)-V_o+\frac{\beta^2t^2}{2!}\right]=t\beta^2\mathtt{Bn}, \tag{4.24}$$

and we have to solve the following Cauchy problem:

$$\begin{gathered}\frac{dt}{d\phi^{(1)}}=\frac{1}{t\beta^2\mathtt{Bn}}\left[U_o\left(\phi^{(1)}\right)-V_o+\frac{\beta^2t^2}{2!}\right],\\ t(\xi_o)=0.\end{gathered} \tag{4.25}$$

We notice that $(4.25)_{(1)}$ is a Bernoulli equation. Therefore, setting $w=t^2$, problem (4.25) becomes

$$\begin{gathered}\frac{dw}{d\phi^{(1)}}=\frac{w}{\mathtt{Bn}}+2\left[\frac{U_o(\phi^{(1)})-V_o}{\beta^2\mathtt{Bn}}\right],\\ w(\xi_o)=0.\end{gathered} \tag{4.26}$$

whose solution is given by

$$t^2\left(\phi^{(1)}\right)=\int_{\xi_o}^{\phi^{(1)}}2\exp\left\{\frac{\phi^{(1)}-s}{\mathtt{Bn}}\right\}\left[\frac{U_o(s)-V_o}{\beta^2\mathtt{Bn}}\right]ds, \tag{4.27}$$

which make sense only if $\phi^{(1)}\leqslant\xi_o$. Integrating (4.27) by parts we get

$$t^2\left(\phi^{(1)}\right)=2\int_{\xi_o}^{\phi^{(1)}}\exp\left\{\frac{\phi^{(1)}-s}{\mathtt{Bn}}\right\}\left[\frac{U_o'(s)}{\beta^2}\right]ds+\frac{2}{\beta^2}\left[V_o-U_o\left(\phi^{(1)}\right)\right]. \tag{4.28}$$

We recall from the previous section that the condition $|\dot{\xi}^{(1)}(t)|<1$ is guaranteed if

$$\frac{dt}{d\phi^{(1)}}<-\frac{1}{2}, \tag{4.29}$$

which, by virtue of(4.25)$_{(1)}$, is equivalent to require that

$$t^2 + \mathrm{Bn}t + \frac{2}{\beta^2}\left(U\left(\phi^{(1)}\right) - V_o\right) < 0. \tag{4.30}$$

Hence, under assumption ($H2$), the discriminant $\Delta = \mathrm{Bn}^2 + 8\beta^{-2}(V_o - U_o(\phi^{(1)})) > \mathrm{Bn}^2 > 0$, and (4.30) is fulfilled when

$$\frac{-\mathrm{Bn} - \sqrt{\Delta}}{2} < 0 \leqslant t < \frac{-\mathrm{Bn} + \sqrt{\Delta}}{2}. \tag{4.31}$$

Therefore, in order to have a unique local solution, we must require that

$$t^2 < \frac{\mathrm{Bn}^2 + \Delta - 2\sqrt{\Delta}\mathrm{Bn}}{4} < \frac{\mathrm{Bn}^2}{2} + \frac{2}{\beta^2}\left[V_o - U_o\left(\phi^{(1)}\right)\right], \tag{4.32}$$

which, exploiting (4.28), becomes

$$2\int_{\xi_o}^{\phi^{(1)}} \exp\left\{\frac{\phi^{(1)} - s}{\mathrm{Bn}}\right\}\left[\frac{U_o'(s)}{\beta^2}\right] ds < \frac{\mathrm{Bn}^2}{2}. \tag{4.33}$$

The latter is automatically satisfied, under assumption ($H3$), recalling that $\phi^{(1)} \leqslant \xi_o$. So, also for the first order we have local uniqueness and existence of the solution $\xi^{(1)}(t) = \phi^{(1)}(t) + t$, where $\phi^{(1)}(t)$ is obtained inverting (4.28).

### 4.3. *i*th-Order Approximation

We now consider here the $i$th-order approximation. The evolution equation of the free boundary is given by (4.12). Proceeding as in the previous sections we set $\phi^{(i)} = \xi^{(i)} - t$, so that (4.12) can be rewritten as

$$\dot{\phi}^{(i)}\left[\frac{\beta^2(i+2)t^{i+1}}{(i+1)!} - \frac{V_o t^{i-1}}{(i-1)!} + \beta^2\int_0^t U^{(i-2)}\left(\phi^{(i)} - t + \theta, \theta\right)d\theta\right] = \frac{t^i}{i!}\beta^2\mathrm{Bn}, \tag{4.34}$$

with $\phi^{(i)}(0) = \xi_o$. Once again we look for $t = t(\phi^{(i)})$, solving this Cauchy problem

$$\begin{gathered}\frac{dt}{d\phi^{(i)}} = \frac{(i+2)}{(i+1)}\frac{t}{\mathrm{Bn}} - \frac{V_o i}{\beta^2\mathrm{Bn}t} + \frac{i!}{t^i\mathrm{Bn}}\int_0^t U^{(i-2)}\left(\phi^{(i)} - t + \theta, \theta\right)d\theta \quad t(\xi_o) = 0, \\ t(\xi_o) = 0.\end{gathered} \tag{4.35}$$

Now, hypothesis ($H4$) and (4.17) entail

$$\lim_{t\to 0^+}\frac{d\phi^{(i)}}{dt} = \lim_{t\to 0^+}\frac{dt}{d\phi^{(i)}} = -1. \tag{4.36}$$

Therefore

$$\frac{V_o i}{\beta^2} = \lim_{t\to 0^+} \frac{i!}{t^{i-1}} \int_0^t U^{(i-2)}\left(\phi^{(i)} - t + \theta, \theta\right) d\theta. \tag{4.37}$$

So for $t$ sufficiently small, we can approximate the integral on the r.h.s. of (4.37) in the following way:

$$\int_0^t U^{(i-2)}\left(\phi^{(i)} - t + \theta, \theta\right) d\theta = C_i\left(\phi^{(i)}\right) t^{i-1}, \tag{4.38}$$

where $C_i$ is a smooth function of $\phi^{(i)}$, determined exploiting (4.7), (4.8), and (4.9). In particular,

$$C_i(\xi_o) = \frac{V_o}{(i-1)!\beta^2 \mathrm{Bn}}. \tag{4.39}$$

Hence, setting

$$A_i = \frac{i+2}{i+1}\frac{1}{\mathrm{Bn}}, \qquad B_i\left(\phi^{(i)}\right) = \left[\frac{i! C_i(\phi^{(i)})}{\mathrm{Bn}} - \frac{i V_o}{\beta^2 \mathrm{Bn}}\right], \tag{4.40}$$

problem (4.35) acquires the following structure:

$$\begin{gathered} \frac{dt}{d\phi^{(i)}} = A_i t + B_i\left(\phi^{(i)}\right)\frac{1}{t}, \quad t(\xi_o) = 0, \\ t(\xi_o) = 0, \end{gathered} \tag{4.41}$$

provided $t$ sufficiently small. From (4.37) it is easy to check that in a right neighborhood of $t = 0$ the function (Recall that $U^{(i-2)}(x,t)$ are everywhere non negative for every $i$), $B_i(\phi^{(i)}) < 0$ and $B_i(\xi_o) = 0$. Equation $(4.41)_{(1)}$ is once again a Bernoulli equation which can be integrated providing

$$t^2\left(\phi^{(i)}\right) = \int_{\xi_o}^{\phi^{(i)}} 2B_i(s) \exp\left\{2A_i\left(\phi^{(i)} - s\right)\right\} ds, \tag{4.42}$$

where we recall once again that $\phi^{(i)} \leqslant \xi_o$. Also in this case we can integrate by parts getting

$$t^2\left(\phi^{(i)}\right) = -\frac{B_i}{A_i} + \int_{\xi_o}^{\phi^{(i)}} 2\frac{B_i'(s)}{A_i} \exp\left\{2A_i\left(\phi^{(i)} - s\right)\right\} ds. \tag{4.43}$$

Proceeding as before we derive the conditions ensuring $|\dot{\xi}^{(i)}| < 1$. Hence, making use of (4.41) we obtain the following ineqaulity:

$$t^2 + \frac{t}{2A_i} + \frac{B_i}{A_i} < 0, \tag{4.44}$$

which is satisfied if

$$t^2 < \frac{1}{8A_i^2} - \frac{B_i}{A_i}. \tag{4.45}$$

Therefore, $|\dot{\xi}^{(i)}| < 1$ when

$$\int_{\xi_o}^{\phi^{(i)}} 2B_i'(s) \exp\left\{2A_i\left(\phi^{(i)} - s\right)\right\} ds < \frac{1}{8A_i}. \tag{4.46}$$

So, if condition (4.46) is fulfilled, the solution is given by $\xi^{(i)}(t) = \phi^{(i)}(t) + t$, where $\phi^{(i)}(t)$ is obtained inverting (4.42).

## 5. Conclusions

We have studied a hyperbolic (telegrapher's equation) free boundary problem derived from the model for a pressure-driven channel flow of a particular Bingham-like fluid described in [1]. The motivation of this analysis comes from the study of the rheology of materials like asphalt and bitumen. Exploiting the representation formulas, determined by means of modified Bessel functions, we have shown that the free boundary equation (which has turned out to be a nonlinear integrodifferential equation) can be rewritten only in terms of the initial and boundary data of the problem. In other words, the free boundary dynamics can be solved autonomously from the problem for the velocity field.

We have shown that local existence and uniqueness is guaranteed under some appropriate assumptions on the initial and boundary data (Theorem 3.3). Moreover, when $\omega < 1$ (and this is the case of asphalt and bitumen), we approximate the solution performing an asymptotic expansion in which each term can be iteratively evaluated. We did not prove the convergence of the asymptotic series.

A further extension of the analysis we have performed in this paper (which is a limit case of the model presented in [1]) would be to study the one-dimensional problem in its general structure, in which the inner part of the layer is treated as an Oldroyd-B fluid (with nonuniform velocity). Of course this problem is by far more complicated than the one described in this paper. Nevertheless the procedure we have employed here seems to be a promising tool.

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