Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2011, Article ID 527434, 9 pages
doi:10.1155/2011/527434

Research Article

Constancy of $\overline{\phi}$-Holomorphic Sectional Curvature for an Indefinite Generalized $g \cdot f \cdot f$-Space Form

Jae Won Lee

Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea

Correspondence should be addressed to Jae Won Lee, leejaewon@sogang.ac.kr

Received 15 June 2011; Revised 18 July 2011; Accepted 16 September 2011

Academic Editor: B. G. Konopelchenko

Bonome et al., 1997, provided an algebraic characterization for an indefinite Sasakian manifold to reduce to a space of constant ϕ-holomorphic sectional curvature. In this present paper, we generalize the same characterization for indefinite $g \cdot f \cdot f$-space forms.

1. Introduction

For an almost Hermitian manifold (M^{2n}, g, J) with $\dim(M) = 2n > 4$, Tanno [1] has proved the following.

Theorem 1.1. *Let* $\dim(M) = 2n > 4$, *and assume that almost Hermitian manifold* (M^{2n}, g, J) *satisfies*

$$R(JX, JY, JZ, JX) = R(X, Y, Z, X) \tag{1.1}$$

for every tangent vector X, Y, and Z. Then (M^{2n}, g, J) has a constant holomorphic sectional curvature at x if and only if

$$R(X, JX)X \text{ is proportional to } JX \tag{1.2}$$

for every tangent vector X at $x \in M$.

Tanno [1] has also proved an analogous theorem for Sasakian manifolds as follows.

Theorem 1.2. *A Sasakian manifold* ≥ 5 *has a constant* ϕ*-sectional curvature if and only if*

$$R(X, \phi X)X \text{ is proportional to } \phi X \tag{1.3}$$

for every tangent vector X *such that* $g(X, \xi) = 0$.

Nagaich [2] has proved the generalized version of Theorem 1.1, for indefinite almost Hermitian manifolds as follows.

Theorem 1.3. *Let* (M^{2n}, g, J) $(n > 2)$ *be an indefinite almost Hermitian manifold that satisfies* (1.1), *then* (M^{2n}, g, J) *has a constant holomorphic sectional curvature at* x *if and only if*

$$R(X, JX)X \text{ is proportional to } JX \tag{1.4}$$

for every tangent vector X *at* $x \in M$.

Bonome et al. [3] generalized Theorem 1.2 for an indefinite Sasakian manifold as follows.

Theorem 1.4. *Let* $(M^{2n+1}, \phi, \eta, \xi, g)$ $(n \geq 2)$ *be an indefinite Sasakian manifold. Then* M^{2n+1} *has a constant* ϕ*-sectional curvature if and only if*

$$R(X, \phi X)X \text{ is proportional to } \phi X \tag{1.5}$$

for every vector field X *such that* $g(X, \xi) = 0$.

In this paper, we generalize Theorem 1.4 for an indefinite generalized $g \cdot f \cdot f$-space form by proving the following.

Theorem 1.5. *Let* $(\overline{M}^{2n+r}, F_1, F_2, F_3)$ *be an indefinite generalized* $g \cdot f \cdot f$*-space form. Then* $\overline{M}^{2n+r}$ *is of constant* ϕ*-sectional curvature if and only if*

$$\overline{R}\left(X, \overline{\phi} X\right)X \text{ is proportional to } \overline{\phi} X \tag{1.6}$$

for every vector field X *such that* $\overline{g}(X, \overline{\xi_\alpha}) = 0$, *for any* $\alpha \in \{1, \ldots, r\}$.

2. Preliminaries

A manifold $\overline{M}$ is called a *globally framed f-manifold (or* $g \cdot f \cdot f$*-manifold)* if it is endowed with a nonnull $(1,1)$-tensor field $\overline{\phi}$ of constant rank, such that $\ker \overline{\phi}$ is parallelizable; that is, there exist global vector fields $\overline{\xi_\alpha}$, $\alpha \in \{1, \ldots, r\}$, with their dual 1-forms $\overline{\eta}^\alpha$, satisfying $\overline{\phi}^2 = -I + \sum_{\alpha=1}^{r} \overline{\eta}^\alpha \otimes \overline{\xi_\alpha}$ and $\overline{\eta}^\alpha(\overline{\xi_\beta}) = \delta^\alpha_\beta$.

The $g\cdot f\cdot f$-manifold $(\overline{M}^{2n+r},\overline{\phi},\overline{\xi}_\alpha,\overline{\eta}^\alpha)$, $\alpha\in\{1,\ldots,r\}$, is said to be an indefinite metric $g\cdot f\cdot f$-manifold if $\overline{g}$ is a semi-Riemannian metric with index ν $(0<\nu<2n+r)$ satisfying the following compatibility condition:

$$\overline{g}\left(\overline{\phi}X,\overline{\phi}Y\right)=\overline{g}(X,Y)-\sum_{\alpha=1}^{r}\epsilon_\alpha\overline{\eta}^\alpha(X)\overline{\eta}^\alpha(Y), \tag{2.1}$$

for any $X,Y\in\Gamma(T\overline{M})$, being $\epsilon_\alpha=\pm1$ according to whether $\overline{\xi}_\alpha$ is spacelike or timelike. Then, for any $\alpha\in\{1,\ldots,r\}$, one has $\overline{\eta}^\alpha(X)=\epsilon_\alpha\overline{g}(X,\overline{\xi}_\alpha)$. Following the notations in [4, 5], we adopt the curvature tensor R, and thus we have $R(X,Y,Z)=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z$ and $\overline{R}(X,Y,Z,W)=\overline{g}(\overline{R}(Z,W,Y),X)$, for any $X,Y,Z,W\in\Gamma(TM)$.

We recall that, as proved in [6], the Levi-Civita connection $\overline{\nabla}$ of an indefinite $g\cdot f\cdot f$-manifold satisfies the following formula:

$$\begin{aligned}2\overline{g}\left(\left(\overline{\nabla}_X\overline{\phi}\right)Y,Z\right)&=3d\Phi\left(X,\overline{\phi}Y,\overline{\phi}Z\right)-3d\Phi(X,Y,Z)\\&\quad+\overline{g}\left(N(Y,Z),\overline{\phi}X\right)+\epsilon_\alpha N_\alpha^{\overline{\phi}}(Y,Z)\overline{\eta}^\alpha(X)\\&\quad+2\epsilon_\alpha d\overline{\eta}^\alpha\left(\overline{\phi}Y,X\right)\overline{\eta}^\alpha(Z)-2\epsilon_\alpha d\overline{\eta}^\alpha\left(\overline{\phi}Z,X\right)\overline{\eta}^\alpha(Y),\end{aligned} \tag{2.2}$$

where $N_\alpha^{\overline{\phi}}$ is given by $N_\alpha^{\overline{\phi}}(X,Y)=2d\overline{\eta}^\alpha(\overline{\phi}X,Y)-2d\overline{\eta}^\alpha(\overline{\phi}Y,X)$.

An indefinite metric $g\cdot f\cdot f$-manifold is called an *indefinite $\mathcal{S}$-manifold* if it is normal and $d\overline{\eta}^\alpha=\Phi$, for any $\alpha\in\{1,\ldots,r\}$, where $\Phi(X,Y)=\overline{g}(X,\overline{\phi}Y)$ for any $X,Y\in\Gamma(T\overline{M})$. The normality condition is expressed by the vanishing of the tensor field $N=N_{\overline{\phi}}+\sum_{\alpha=1}^{r}2d\overline{\eta}^\alpha\otimes\overline{\xi}_\alpha$, $N_{\overline{\phi}}$ being the Nijenhuis torsion of $\overline{\phi}$.

Furthermore, the Levi-Civita connection of an indefinite $\mathcal{S}$-manifold satisfies

$$\left(\overline{\nabla}_X\overline{\phi}\right)Y=\overline{g}\left(\overline{\phi}X,\overline{\phi}Y\right)\overline{\xi}+\overline{\eta}(Y)\overline{\phi}^2(X), \tag{2.3}$$

where $\overline{\xi}=\sum_{\alpha=1}^{r}\overline{\xi}_\alpha$ and $\overline{\eta}=\sum_{\alpha=1}^{r}\epsilon_\alpha\overline{\eta}^\alpha$. We recall that $\overline{\nabla}_X\overline{\xi}_\alpha=-\epsilon_\alpha\overline{\phi}X$ and $\ker\overline{\phi}$ is an integrable flat distribution since $\overline{\nabla}_{\overline{\xi}_\alpha}\overline{\xi}_\beta=0$ (see more details in [6]).

A plane section in $T_p\overline{M}$ is a $\overline{\phi}$-holomorphic section if there exists a vector $X\in T_p\overline{M}$ orthogonal to $\overline{\xi}_1,\ldots,\overline{\xi}_r$ such that $\{X,\overline{\phi}X\}$ span the section. The sectional curvature of a $\overline{\phi}$-holomorphic section, denoted by $c(X)=R(X,\overline{\phi}X,\overline{\phi}X,X)$, is called a $\overline{\phi}$-holomorphic sectional curvature.

Proposition 2.1 (see [7]). *An indefinite Sasakian manifold* $(\overline{M}^{2n+1}, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ *has* $\overline{\phi}$*-sectional curvature c if and only if its curvature tensor verifies*

$$\begin{aligned}\overline{R}(X,Y)Z = {} & \frac{(c+3)}{4}\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\} \\ & + \frac{(c-1)}{4}\Big\{\Phi(X,Z)\overline{\phi}Y - \Phi(Y,Z)\overline{\phi}X + 2\Phi(X,Y)\ \overline{\phi}Z \\ & - \overline{g}(Z,Y)\overline{\eta}(X)\overline{\xi} + \overline{g}(Z,X)\overline{\eta}(Y)\overline{\xi} - \overline{\eta}(Y)\overline{\eta}(Z)X + \overline{\eta}(Z)\overline{\eta}(X)Y\Big\}\end{aligned} \tag{2.4}$$

for any vector fields $X, Y, Z, W \in \Gamma(T\overline{M})$.

A Sasakian manifold $\overline{M}^{2n+1}$ *with constant* $\overline{\phi}$*-sectional curvature* $c \in \mathbb{R}$ *is called a Sasakian space form, denoted by* $\overline{M}^{2n+1}(c)$.

Definition 2.2. An almost contact metric manifold $(\overline{M}^{2n+1}, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is an *indefinite generalized Sasakian space form*, denoted by $\overline{M}^{2n+1}(f_1, f_2,$ and $f_3)$, if it admits three smooth functions f_1, f_2, f_3 such that its curvature tensor field verifies

$$\begin{aligned}\overline{R}(X,Y)Z = {} & f_1\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\} \\ & + f_2\Big\{\Phi(X,Z)\overline{\phi}Y - \Phi(Y,Z)\overline{\phi}X + 2\Phi(X,Y)\overline{\phi}Z\Big\} \\ & + f_3\Big\{-\overline{g}(Z,Y)\overline{\eta}(X)\overline{\xi} + \overline{g}(Z,X)\overline{\eta}(Y)\overline{\xi} \\ & - \overline{\eta}(Y)\overline{\eta}(Z)X + \overline{\eta}(Z)\overline{\eta}(X)Y\Big\}\end{aligned} \tag{2.5}$$

for any vector fields $X, Y, Z, W \in \Gamma(T\overline{M})$.

Remark 2.3. Any indefinite generalized Sasakian space form has $\overline{\phi}$-sectional curvature $c = f_1 + 3f_2$. Indeed, $f_1 = (c+3)/4$ and $f_2 = f_3 = (c-1)/4$.

Proposition 2.4 (see [6]). *An indefinite* $\mathcal{S}$*-manifold* $\overline{M}^{2n+r}$ *has* $\overline{\phi}$*-sectional curvature c if and only if its curvature tensor verifies*

$$\begin{aligned}\overline{R}(X,Y)Z = {} & \frac{(c+3\epsilon)}{4}\Big\{\overline{g}\big(\overline{\phi}X,\overline{\phi}Z\big)\overline{\phi}^2Y - \overline{g}\big(\overline{\phi}Y,\overline{\phi}Z\big)\overline{\phi}^2X\Big\} \\ & + \frac{(c-\epsilon)}{4}\Big\{\Phi(Z,Y)\overline{\phi}X - \Phi(Z,X)\overline{\phi}Y + 2\Phi(X,Y)\overline{\phi}Z\Big\} \\ & + \Big\{\overline{\eta}(Z)\overline{\eta}(X)\overline{\phi}^2Y - \overline{\eta}(Y)\overline{\eta}(Z)\overline{\phi}^2X + \overline{g}\big(\overline{\phi}Z,\overline{\phi}Y\big)\overline{\eta}(X)\overline{\xi} - \overline{g}\big(\overline{\phi}Z,\overline{\phi}X\big)\overline{\eta}(Y)\overline{\xi}\Big\}\end{aligned} \tag{2.6}$$

for any vector fields $X, Y, Z, W \in \Gamma(T\overline{M})$ *and* $\epsilon = \sum \epsilon_\alpha$.

An indefinite $\mathcal{S}$-manifold $\overline{M}^{2n+r}$ with constant $\overline{\phi}$-sectional curvature $c \in \mathbb{R}$ is called a $\mathcal{S}$-space form, denoted by $\overline{M}^{2n+r}(c)$. One remarks that for $r = 1$ (2.6) reduces to (2.4).

3. An Indefinite Generalized $g \cdot f \cdot f$-Manifold

Let $\digamma$ denote any set of smooth functions F_{ij} on $\overline{M}^{2n+r}$ such that $F_{ij} = F_{ji}$ for any $i, j \in \{1, \dots, r\}$.

Definition 3.1. An *indefinite generalized $g \cdot f \cdot f$-space-form*, denoted by $(\overline{M}^{2n+r}, F_1, F_2, \digamma)$, is an indefinite $g \cdot f \cdot f$-manifold $(\overline{M}^{2n+r}, \overline{\phi}, \overline{\xi_\alpha}, \overline{\eta}^\alpha, \overline{g})$ which admits smooth function $F_1, F_2, \digamma$ such that its curvature tensor field verifies

$$\begin{aligned}
\overline{R}(X,Y)Z = {} & F_1\left\{\overline{g}\left(\overline{\phi}X,\overline{\phi}Z\right)\overline{\phi}^2 Y - \overline{g}\left(\overline{\phi}Y,\overline{\phi}Z\right)\overline{\phi}^2 X\right\} \\
& + F_2\left\{\Phi(Z,Y)\overline{\phi}X - \Phi(Z,X)\overline{\phi}Y + 2\Phi(X,Y)\overline{\phi}Z\right\} \\
& + \sum_{\alpha,\beta=1}^{r} F_{\alpha\beta}\left\{\overline{\eta}^\alpha(X)\overline{\eta}^\beta(Z)\overline{\phi}^2 Y - \overline{\eta}^\alpha(Y)\overline{\eta}^\beta(Z)\overline{\phi}^2 X\right. \\
& \left. + \overline{g}\left(\overline{\phi}Z,\overline{\phi}Y\right)\overline{\eta}^\alpha(X)\overline{\xi_\beta} - \overline{g}\left(\overline{\phi}Z,\overline{\phi}X\right)\overline{\eta}^\alpha(Y)\overline{\xi_\beta}\right\}
\end{aligned} \tag{3.1}$$

for any vector fields $X, Y, Z, W \in \Gamma(T\overline{M})$.

For $r = 1$, we obtain an indefinite Sasakian space form $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with $f_1 = F_1$, $f_2 = F_2$, and $f_3 = F_1 - F_{11}$. In particular, if the given structure is Sasakian, (3.1) holds with $F_{11} = 1$, $F_1 = (c+3)/4$, $F_3 = (c-1)/4$, and $f_3 = F_1 - F_{11} = (c-1)/4 = f_2$.

Theorem 3.2. *Let $(\overline{M}^{2n+r}, F_1, F_2, \digamma)$ be an indefinite generalized $g \cdot f \cdot f$-space form. Then $\overline{M}^{2n+r}$ is of constant ϕ-sectional curvature if and only if*

$$\overline{R}\left(X,\overline{\phi}X\right)X \text{ is proportional to } \overline{\phi}X \tag{3.2}$$

for every vector field X such that $\overline{g}(X, \overline{\xi_\alpha}) = 0$, for any $\alpha \in \{1, \dots, r\}$.

Proof. Let $(\overline{M}^{2n+r}, F_1, F_2, \digamma)$ be an indefinite generalized $g \cdot f \cdot f$-space form. To prove the theorem for $n \geq 2$, we will consider cases when $n = 2$ and when $n > 2$, that is, when $n \geq 3$.

Case 1 $(\overline{g}(X,X) = \overline{g}(Y,Y))$. The proof is similar as given by Lee and Jin [8], so we drop the proof.

Case 2 $(\overline{g}(X,X) = -\overline{g}(Y,Y))$. Here, if X is spacelike, then Y is timelike or vice versa. First of all, assume that $\overline{M}$ is of constant $\overline{\phi}$-holomorphic sectional curvature. Then (3.1) gives

$$\overline{R}\left(X,\overline{\phi}X\right)X = \{F_1 + 3F_2\}\overline{\phi}X = c\overline{\phi}X. \tag{3.3}$$

Conversely, let $\{X, Y\}$ be an orthonormal pair of tangent vectors such that $\overline{g}(\overline{\phi}X, Y) = \overline{g}(X, Y) = \overline{g}(Y, \overline{\xi_\alpha}) = 0$, $\alpha \in \{1, \dots, r\}$, and $n \geq 3$. Then $\ddot{X} = (X+iY)/\sqrt{2}$ and $\ddot{Y} = (i\overline{\phi}X+\overline{\phi}Y)/\sqrt{2}$ also form an orthonormal pair of tangent vectors such that $\overline{g}(\overline{\phi}\ddot{X}, \ddot{Y}) = 0$. Then (3.1) and curvature properties give

$$\begin{aligned} 0 &= \overline{R}\left(\ddot{X}, \overline{\phi}\ddot{X}, \ddot{Y}, \ddot{X}\right) \\ &= \overline{g}\left(\overline{R}\left(X, \overline{\phi}X\right)X, \overline{\phi}X\right) - \overline{g}\left(\overline{R}\left(Y, \overline{\phi}Y\right)Y, \overline{\phi}Y\right) \\ &\quad - 2\overline{g}\left(\overline{R}\left(X, \overline{\phi}Y\right)Y, \overline{\phi}Y\right) + 2\overline{g}\left(\overline{R}\left(X, \overline{\phi}X\right)Y, \overline{\phi}X\right). \end{aligned} \tag{3.4}$$

From the assumption, we see that the last two terms of the right-hand side vanish. Therefore, we get $c(X) = c(Y)$.

Now, if $\operatorname{span}\{U, V\}$ is $\overline{\phi}$-holomorphic, then for $\overline{\phi}U = aU + bV$, where a and b are constant, we have

$$\operatorname{span}\left\{U, \overline{\phi}U\right\} = \operatorname{span}\{U, aU + bV\} = \operatorname{span}\{U, V\}. \tag{3.5}$$

Similarly,

$$\operatorname{span}\left\{V, \overline{\phi}V\right\} = \operatorname{span}\{U, V\}, \qquad \operatorname{span}\left\{U, \overline{\phi}U\right\} = \operatorname{span}\left\{V, \overline{\phi}V\right\}. \tag{3.6}$$

These imply

$$\overline{R}\left(U, \overline{\phi}U, U, \overline{\phi}U\right) = \overline{R}\left(V, \overline{\phi}V, V, \overline{\phi}V\right), \quad \text{or} \quad c(U) = c(V). \tag{3.7}$$

If $\operatorname{span}\{U, V\}$ is not $\overline{\phi}$-holomorphic section, then we can choose unit vectors $X \in \operatorname{span}\{U, \overline{\phi}U\}^\perp$ and $Y \in \operatorname{span}\{V, \overline{\phi}V\}^\perp$ such that $\operatorname{span}\{X, Y\}$ is $\overline{\phi}$-holomorphic. Thus we get

$$c(U) = c(X) = c(Y) = c(V), \tag{3.8}$$

which shows that any $\overline{\phi}$-holomorphic section has the same $\overline{\phi}$-holomorphic sectional curvature.

Now, let $n = 2$, and let $\{X, Y\}$ be a set of orthonormal vectors such that $\overline{g}(X, X) = -\overline{g}(Y, Y)$ and $\overline{g}(X, \overline{\phi}X) = 0$, and we have $c(X) = c(Y)$ as before. Using the property (3.2), we get

$$\begin{aligned}
\overline{R}\left(X, \overline{\phi}X\right)X &= -\{F_1 + 3F_2\}\overline{\phi}X = -c(X)\overline{\phi}X, \\
\overline{R}\left(X, \overline{\phi}X\right)Y &= -2F_2\overline{\phi}Y, \\
\overline{R}\left(X, \overline{\phi}Y\right)X &= -F_1\overline{\phi}Y, \\
\overline{R}\left(X, \overline{\phi}Y\right)Y &= F_2\overline{\phi}X, \\
\overline{R}\left(Y, \overline{\phi}X\right)Y &= F_1\overline{\phi}X, \\
\overline{R}\left(Y, \overline{\phi}X\right)X &= -F_2\overline{\phi}Y, \\
\overline{R}\left(Y, \overline{\phi}Y\right)X &= 2F_2\overline{\phi}X, \\
\overline{R}\left(Y, \overline{\phi}Y\right)Y &= \{F_1 + 3F_2\}\overline{\phi} = c(Y)\overline{\phi}Y = c(X)\overline{\phi}Y.
\end{aligned} \tag{3.9}$$

Now, define $\widehat{X} = aX + bY$ such that $a^2 - b^2 = 1$ and $a^2 \neq b^2$. Using the above relations, we get

$$R\left(\widehat{X}, \overline{\phi}\widehat{X}\right)\widehat{X} = C_1\overline{\phi}X + C_2\overline{\phi}Y. \tag{3.10}$$

Therefore, we have

$$\begin{aligned}
C_1 &= -a^3c(X) + ab^2c(X), \\
C_2 &= b^3c(X) - a^2bc(X).
\end{aligned} \tag{3.11}$$

On the other hand,

$$\overline{R}\left(\widehat{X}, \overline{\phi}\widehat{X}\right)\widehat{X} = c\left(\widehat{X}\right)\overline{\phi}\widehat{X} = c\left(\widehat{X}\right)\left\{a\overline{\phi}X + b\overline{\phi}Y\right\}. \tag{3.12}$$

Comparing (3.11) and (3.12), we get

$$\begin{aligned}
-a^2c(X) + b^2c(X) &= c\left(\widehat{X}\right), \\
b^2c(X) - a^2c(X) &= c\left(\widehat{X}\right).
\end{aligned} \tag{3.13}$$

On solving (3.13), we have

$$c(X) = c\left(\widehat{X}\right). \tag{3.14}$$

Similary, we can prove

$$c(Y) = c\left(\widehat{Y}\right). \tag{3.15}$$

Therefore, $\overline{M}$ has constant $\overline{\phi}$-holomorphic sectional curvature.

Case 3 ($\overline{g}(U,U) = 0$). It is enough to show a sufficient condition. Let Y_α be a unit vector tangent to $\overline{\xi}_\alpha$, for any $\alpha \in \{1,\ldots,r\}$, such that $\overline{g}(Y_\alpha, Y_\alpha) = -\overline{g}(\xi_\alpha, \xi_\alpha) = -\epsilon_\alpha$, and consider the null vector $U_\alpha = \xi_\alpha + Y$. From (3.2),

$$\begin{aligned} c(U_\alpha)\overline{\phi}U_\alpha &= c(U_\alpha)\overline{\phi}(\xi_\alpha + Y_\alpha) \\ &= \overline{R}\left(\xi_\alpha + Y_\alpha, \overline{\phi}(\xi_\alpha + Y_\alpha)\right)(\xi_\alpha + Y_\alpha). \end{aligned} \tag{3.16}$$

Therefore,

$$\begin{aligned} c(U_\alpha) &= \overline{g}\left(c(U_\alpha)\overline{\phi}(\xi_\alpha + Y_\alpha), \epsilon_\alpha\overline{\phi}Y_\alpha\right) \\ &= \epsilon_\alpha\overline{g}\left(\overline{R}\left(\xi_\alpha + Y_\alpha, \overline{\phi}(\xi_\alpha + Y_\alpha)\right)(\xi_\alpha + Y_\alpha), \overline{\phi}Y_\alpha\right) \\ &= \epsilon_\alpha\overline{g}\left(\overline{R}\left(\xi_\alpha, \overline{\phi}\xi_\alpha\right)\xi_\alpha, \overline{\phi}Y_\alpha\right) + \epsilon_\alpha\overline{g}\left(\overline{R}\left(\xi_\alpha, \overline{\phi}Y_\alpha\right)\xi_\alpha, \overline{\phi}Y_\alpha\right) \\ &\quad + \epsilon_\alpha\overline{g}\left(\overline{R}\left(\xi_\alpha, \overline{\phi}\xi_\alpha\right)Y_\alpha, \overline{\phi}Y_\alpha\right) + \epsilon_\alpha\overline{g}\left(\overline{R}\left(\xi_\alpha, \overline{\phi}Y_\alpha\right)Y_\alpha, \overline{\phi}Y_\alpha\right) \\ &\quad + \epsilon_\alpha\overline{g}\left(\overline{R}\left(Y_\alpha, \overline{\phi}\xi_\alpha\right)\xi_\alpha, \overline{\phi}Y_\alpha\right) + \epsilon_\alpha\overline{g}\left(\overline{R}\left(Y_\alpha, \overline{\phi}Y_\alpha\right)\xi_\alpha, \overline{\phi}Y_\alpha\right) \\ &\quad + \epsilon_\alpha\overline{g}\left(\overline{R}\left(Y_\alpha, \overline{\phi}\xi_\alpha\right)Y_\alpha, \overline{\phi}Y_\alpha\right) + \epsilon_\alpha\overline{g}\left(\overline{R}\left(Y_\alpha, \overline{\phi}Y_\alpha\right)Y_\alpha, \overline{\phi}Y_\alpha\right) \\ &\quad + \epsilon_\alpha\overline{g}\left(\overline{R}\left(\xi_\alpha, \overline{\phi}Y_\alpha\right)\xi_\alpha, \overline{\phi}Y_\alpha\right) + 2\epsilon_\alpha\overline{g}\left(\overline{R}\left(Y_\alpha, \overline{\phi}Y_\alpha\right)\xi_\alpha, \overline{\phi}Y_\alpha\right) \\ &\quad + \epsilon_\alpha\overline{g}\left(\overline{R}\left(Y_\alpha, \overline{\phi}Y_\alpha\right)Y_\alpha, \overline{\phi}Y_\alpha\right) \\ &= \epsilon_\alpha\overline{g}\left(\overline{R}\left(Y_\alpha, \overline{\phi}Y_\alpha\right)Y_\alpha, \overline{\phi}Y_\alpha\right). \end{aligned} \tag{3.17}$$

From Cases 1 and 2, depending on the sign of ϵ_α, $\overline{g}(\overline{R}(Y_\alpha, \overline{\phi}Y_\alpha)Y_\alpha, \ \overline{\phi}Y_\alpha) = \epsilon_\alpha c(Y_\alpha)$ is constant, and hence $c(U_\alpha) = c(Y_\alpha)$ is constant. □

Theorem 3.3 (see [9]). *Let* $(\overline{M}^{2n+r}, \overline{\phi}, \overline{\eta}^{\alpha}, \overline{\xi_\alpha}, \overline{g})(n \geq 2)$ *be an indefinite* $\mathcal{S}$*-manifold. Then* M^{2n+r} *is of constant* ϕ*-sectional curvature if and only if*

$$R\Big(X, \overline{\phi}X\Big)X \text{ is proportional to } \overline{\phi}X \tag{3.18}$$

for every vector field X *such that* $g(X, \overline{\xi_\alpha}) = 0$*, for any* $\alpha \in \{1, \ldots, r\}$*.*

Proof. An $\mathcal{S}$-space form is a special case of $g \cdot f \cdot f$-space form, and hence the proof follows from Theorem 3.2 and (2.6). □

Theorem 3.4 (cf. Bonome et al. [3]). *Let* $(M^{2n+1}, \phi, \eta, \xi, g)(n \geq 2)$ *be an indefinite Sasakian manifold. Then* M^{2n+1} *is of constant* ϕ*-sectional curvature if and only if*

$$R(X, \phi X)X \text{ is proportional to } \phi X \tag{3.19}$$

for every vector field X *such that* $g(X, \xi) = 0$*.*

Proof. When $r = 1$, an indefinite $\mathcal{S}$-space form $M^{2n+1}(c)$ reduces to a Sasakian space form. The proof follows from (2.4) and Theorem 3.3. □

References

[1] S. Tanno, "Constancy of holomorphic sectional curvature in almost Hermitian manifolds," *Kodai Mathematical Seminar Reports*, vol. 25, pp. 190–201, 1973.

[2] R. K. Nagaich, "Constancy of holomorphic sectional curvature in indefinite almost Hermitian manifolds," *Kodai Mathematical Journal*, vol. 16, no. 2, pp. 327–331, 1993.

[3] A. Bonome, R. Castro, E. Garcia-Rio, and L. Hervella, "Curvature of indefinite almost contact manifolds," *Journal of Geometry*, vol. 58, no. 1-2, pp. 66–86, 1997.

[4] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. 1, Interscience Publishers, New York, NY, USA, 1963.

[5] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. 2, Interscience Publishers, New York, NY, USA, 1969.

[6] L. Brunetti and A. M. Pastore, "Curvature of a class of indefinite globally framed f-manifolds," *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 51(99), no. 3, pp. 183–204, 2008.

[7] T. Ikawa and J. B. Jun, "On sectional curvatures of normal contact Lorentzian manifold," *Journal of the Korean Mathematical Society*, vol. 4, pp. 27–33, 1997.

[8] J. W. Lee and D. H. Jin, "Constancy of ϕ—holomorphic scetional curvature in generalized $g \cdot f \cdot f$— manifolds ," http://arxiv.org/abs/1103.5266v1 .

[9] J. W. Lee, "Constancy of ϕ-sectional curvature in indefinite S-manifolds," *British Journal of Mathematics and Computer Science*, vol. 1, no. 3, pp. 121–128, 2011.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization