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Research Article

On the Exact Solution of a Generalized Polya Process

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There are two types of master equations in describing nonequilibrium phenomena with memory effect: (i) the memory function type and (ii) the nonstationary type. A generalized Polya process is studied within the framework of a non-stationary type master equation approach. For a transition-rate with an arbitrary time-dependent relaxation function, the exact solution of a generalized Polya process is obtained. The characteristic features of temporal variation of the solution are displayed for some typical time-dependent relaxation functions reflecting memory in the systems.

1. Introduction

The generalized master equation of memory function type [1] is a useful basis for analyzing non-equilibrium phenomena in open systems as

$$\frac{d}{dt}P(n,t) = \int_0^t d\tau \sum_j \left[K_{nj}(t-\tau)P(j,\tau) - K_{jn}(t-\tau)P(n,\tau)\right], \tag{1.1}$$

where the kernel $K_{nj}(t)$ is conventionally assumed to have the product of a memory function $\phi(t)$ with a transition rate $w(n,j)$ as $K_{nj}(t) = \phi(t)w(n,j)$. The transition rate $w(n,j)$ has the constraint with $\sum_j w(j,n) = 1$. This generalized master equation approach corresponds to the generalized Langevin equation of the memory function type [2, 3]. One can see many successful applications with long memory along the line of traditional formulation [1].

Looking around recent studies in complex open systems, there is an alternative approach based on a generalized non-stationary master equation [4] as

$$\frac{d}{dt}P(n,t) = \sum_j \left[L_{nj}(t)P(j,t) - L_{jn}(t)P(n,t)\right]. \tag{1.2}$$

The master equation in this form corresponds to the generalized Langevin equation of the convolutionless type, which is derived with the aid of projection operator method by Tokuyama and Mori [5]. The time-dependent coefficient $L_{jk}(t)$ may be written in the following form: $L_{nj}(t) = \phi_c(t)w(n,j)$. It is expected from the projection operator method [5] that the time-dependent function $\phi_c(t)$ reflects the memory effect from varying environment in a different way associated with the memory function (cf. also Hänggi and Talkner [6]). The memory function (MF) formalism has been utilized in anomalous diffusion like Lévy type diffusion in atmospheric pollution, diffusion impurities in amorphous materials, and so on. The alternative convolution-less, non-stationary (NS) formalism gives only a small number of applications. The paper intends to exhibit a potential ability of the NS formalism by taking an arbitrary time-dependent function $\phi_c(t)$ which is representing memory effect.

The paper is organized as follows. Section 2 reviews the non-stationary Poisson process. Section 3 shows a generalized Polya process wherein it is involved a generalized non-stationary transition rate $\lambda(n,t) = \kappa(t)(\alpha n + \beta)$ with an arbitrary function $\kappa(t)$ of time. The exact solution and the expression mean and variance are displayed as a function of $\kappa(t)$. Some important remarks are given for a generalized non-stationary Yule-Furrey process with $\lambda(n,t) = \kappa(t)(\alpha n)$. Section 4 discusses (i) the solvability condition of the generalized Polya model and (ii) the relation to the memory function approach. The last section is devoted to concluding remarks.

2. Nonstationary Poisson Process

The simplest example of the generalized master equation in the form of (1.2) is a non-stationary Poisson process (an inhomogeneous Poisson process) described by

$$\begin{aligned} \frac{d}{dt}p(n,t) &= \lambda(t)p(n-1,t) - \lambda(t)p(n,t) \quad (n \geq 1), \\ \frac{d}{dt}p(0,t) &= -\lambda(t)p(0,t) \quad (n = 0), \end{aligned} \tag{2.1}$$

where $\lambda(t)$ is the time-dependent rate of occurrence of an event $[\lambda(t) \equiv \phi_c(t)w(n-1,n)]$. The function $\lambda(t)$ is an arbitrary function of time. The solution is readily obtained, with the aid of the generating function, in the following form:

$$p(n,t) = \frac{(\Lambda(t))^n}{n!}\exp(-\Lambda(t)), \tag{2.2}$$

Table 1: Some typical examples of $\lambda(t)$ and $\Lambda(t)$.

	$\lambda(t)$	$\langle n(t)\rangle = \sigma_n^2(t) = \Lambda(t)$	constraint
(i)	$\lambda_0 \exp(-\gamma_0 t)$	$\dfrac{\lambda_0}{\gamma_0}(1-\exp(-\gamma_0 t))$	$\gamma_0 > 0$
(ii)	$\dfrac{\lambda_0}{1+\gamma_0 t}$	$\dfrac{\lambda_0}{\gamma_0}\ln(1+\gamma_0 t)$	$\gamma_0 > 0$
(iii)	$\lambda_0 t^{\gamma_0 - 1}$	$\dfrac{\lambda_0}{\gamma_0} t^{\gamma_0}$	$0 < \gamma_0 < 1$

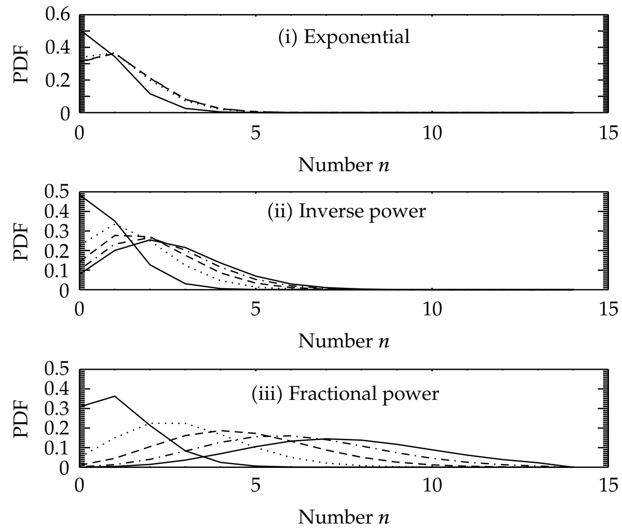


Figure 1: Time dependence of $p(n,t)$ in (2.2) for three relaxation functions of $\lambda(t)$ in Table 1: (i) an exponential function, (ii) an inverse power function, and (iii) a fractional power function; the values of parameters are $\lambda_0 = 1, \gamma_0 = 0.85$. The pdf profiles are depicted for $t = 1$ (solid line), $t = 3$ (dotted line), $t = 5$ (dashed line), $t = 7$ (dash-dotted line), and $t = 9$ (dash-dotted line).

where $\Lambda(t) = \int_0^t d\tau \lambda(\tau)$. It is easy to show that the mean and the variance take the same value $\langle n(t)\rangle = \sigma_n^2(t) = \Lambda(t)$. Namely, the Fano factor $F(= \sigma_n^2(t)/\langle n(t)\rangle)$ takes 1 for any time-dependent function $\lambda(t)$. The process gives rise to only the Poissonian (P) statistics at any time.

Three typical examples of $\lambda(t)$ are shown in Table 1. All of them are relaxation functions $\lambda(t) \to 0$ as the time t goes to infinity. It is shown in the same table that $\Lambda(t) \equiv \int_0^t \lambda(s)ds$ is the increasing function as the time goes to infinity. The temporal development of the probability density $p(n,t)$ in (2.2) for these three examples is depicted in Figure 1.

In seismology, $\lambda(t) = \lambda_0/(1+\alpha t)$ (Ohmori formula) [7] is frequently used in analyzing and predicting aftershocks. Many applications are also found in environmental, insurance, and financial problems [8]. Further, various engineering problems involve many potential applications especially in the probabilistic risk analysis [9]. However, the applicability of the non-stationary Poisson process is quite limited since $\langle n(t)\rangle = \sigma_n^2(t)$.

3. Generalized Polya Process

3.1. Model Equation

Now let us consider a generalized Polya process within the class of generalized birth processes

$$\frac{d}{dt}p(n,t) = \lambda(n-1,t)p(n-1,t) - \lambda(n,t)p(n,t) \quad (n \geq 1), \tag{3.1}$$

$$\frac{d}{dt}p(0,t) = -\lambda(0,t)p(0,t) \quad (n = 0), \tag{3.2}$$

where $\lambda(n,t)$ takes into account the n-dependence up to the first order and a memory effect with an arbitrary relaxation function $\kappa(t)$ as

$$\lambda(n,t) = \kappa(t)\left(\alpha n + \beta\right). \tag{3.3}$$

When $\kappa(t) = 1/(1+\alpha t)$ and $\beta = 1$, the model reduces to a Polya process [10]. When $\kappa(t) = \kappa_0/(1+\gamma_0 t)$, the model reduces to an extended Polya process [11].

3.2. Exact Solution

The method of characteristic curves is used to get the exact solution under the initial condition $p(n,0) = \delta_{n,n_0}$ (cf., the recursion method with variable transformations [11]). The generating function is defined by $g(z,t) = \sum_{n=0}^{\infty} z^n p(n,t)$. The equation for $g(z,t)$ corresponding to (3.1) becomes

$$\frac{d}{dt}g(z,t) = \kappa(t)(z-1)\left\{\alpha z\frac{\partial}{\partial z}g(z,t) + \beta g(z,t)\right\}. \tag{3.4}$$

From the initial condition, one obtains $g(z,0) = z^{n_0}$. To eliminate the second term in the right hand side of (3.4), let us assume that

$$g(z,t) = C(z)G(z,t), \tag{3.5}$$

when $\alpha z(d/dz)C(z) + \beta C(z) = 0$, one obtains $C(z) = C_0 z^{-\beta/\alpha}$. Without the loss of generality, $C_0 = 1$. So the equation for $G(z,t)$ becomes

$$\frac{\partial}{\partial t}G(z,t) = \kappa(t)\alpha z(z-1)\frac{\partial}{\partial z}G(z,t). \tag{3.6}$$

Then, a variable transformation,

$$\xi = \frac{1}{\alpha}\int \frac{1}{z(z-1)}dz = \frac{1}{\alpha}\ln\left(\frac{z-1}{z}\right), \tag{3.7}$$

leads (3.6) to the simple wave equation,

$$\frac{\partial}{\partial t}G(\xi,t) = \kappa(t)\frac{\partial}{\partial \xi}G(\xi,t). \tag{3.8}$$

The solution of the wave equation in (3.8) is given by

$$G(\xi,t) = f(\xi + K(t)), \tag{3.9}$$

where $K(t) = \int_0^t \kappa(\tau)d\tau$. From the initial condition $p(n,0) = \delta_{n,n_0}$, one obtains

$$g(z,0) = z^{-\beta/\alpha} f\left(\frac{1}{\alpha}\ln\frac{z-1}{z}\right) = z^{n_0}. \tag{3.10}$$

Therefore, $f(x)$ is expressed as

$$f(x) = \left(\frac{1}{1-\exp(\alpha x)}\right)^{\beta/\alpha+n_0}. \tag{3.11}$$

Thus, we have

$$g(z,t) = z^{-\beta/\alpha}G(z,t), \tag{3.12}$$

where

$$G(z,t) = \left(\frac{z\exp(-\Lambda(t))}{1-z(1-\exp(-\Lambda(t)))}\right)^{\beta/\alpha+n_0}, \tag{3.13}$$

and $\Lambda(t) = \alpha K(t)$. When $n_0 = 0$, the exact analytic expression of the probability density function $p(n,t)$ is given by

$$p(n,t) = (\exp(-\Lambda(t)))^{\beta/\alpha}\binom{-\frac{\beta}{\alpha}}{n}(-1)^n(1-\exp(-\Lambda(t)))^n \tag{3.14}$$

$$= (\exp(-\Lambda(t)))^{\beta/\alpha}\binom{n+\frac{\beta}{\alpha}-1}{n}(1-\exp(-\Lambda(t)))^n. \tag{3.15}$$

Table 2: Some examples of $\kappa(t)$, and corresponding $\Lambda(t)$ and $\exp(\Lambda(t))$.

	$\kappa(t)$	$\Lambda(t)$	$F \equiv \exp(\Lambda(t))$	constraint
(i)	$\kappa \exp(-\gamma t)$	$\frac{\alpha\kappa}{\gamma}(1-\exp(-\gamma t))$	$\exp\left(\frac{\alpha\kappa}{\gamma}(1-\exp(-\gamma t))\right)$	$\gamma > 0$
(ii)	$\frac{\kappa}{1+\gamma t}$	$\frac{\alpha\kappa}{\gamma}\ln(1+\gamma t)$	$(1+\gamma t)^{\alpha\kappa/\gamma}$	$\gamma > 0$
(iii)	$\kappa t^{\gamma-1}$	$\frac{\alpha\kappa}{\gamma}t^{\gamma}$	$\exp\left(\frac{\alpha\kappa}{\gamma}t^{\gamma}\right)$	$0 < \gamma < 1$

3.3. Mean and Variance

The probability density function in (3.15) is the Pascal distribution $f(\theta)$ (the negative binomial distribution)

$$f(\theta) = \binom{n+r-1}{n}\theta^{r}(1-\theta)^{n}, \tag{3.16}$$

with the parameters (r, θ) with $r = \beta/\alpha$ and $\theta = \exp(-\Lambda(t))$. Thus, the mean and the variance are obtained in the following form:

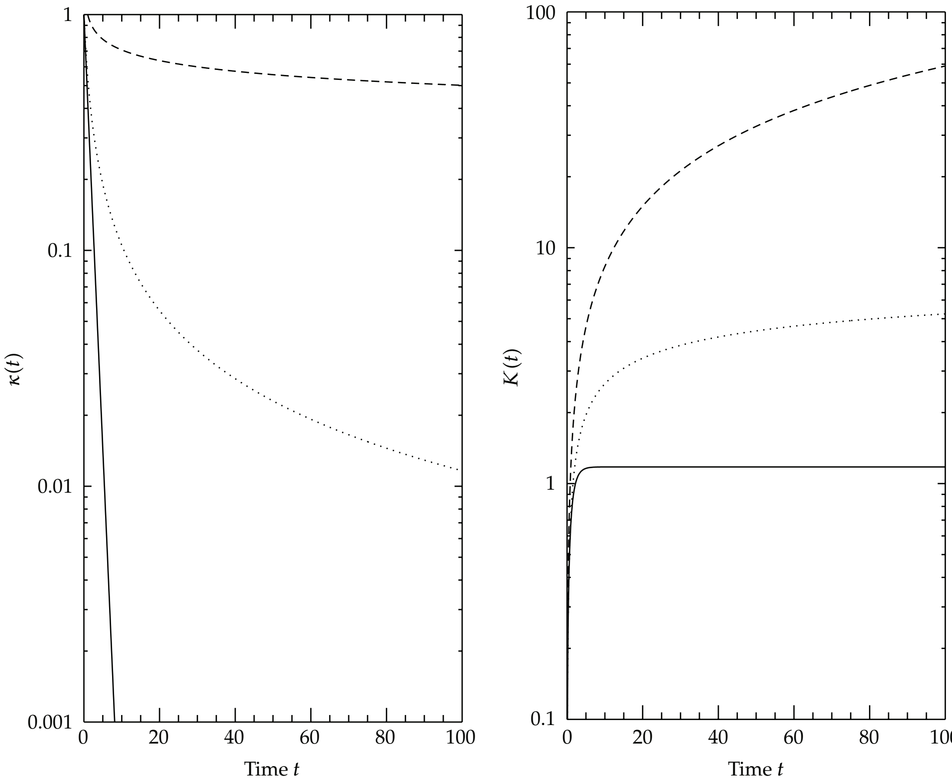
$$\langle n(t)\rangle = \frac{\beta}{\alpha}\left[\exp(\Lambda(t)) - 1\right], \qquad V(t) \equiv \sigma_n^2(t) = \frac{\beta}{\alpha}\exp(\Lambda(t))\left[\exp(\Lambda(t)) - 1\right]. \tag{3.17}$$

The variance is generally greater than the mean, that is, the Fano factor is larger than 1 as follows:

$$F(t) = \frac{V(t)}{\langle n(t)\rangle} = \exp(\Lambda(t)) > 1. \tag{3.18}$$

It is shown that the generalized Polya process with $\alpha \neq 0$ and $\beta \neq 0$ is subjected to the super-Poissonian (SUPP) statistics ($F > 1$).

Three examples of the relaxation function $\kappa(t)$ are given in Table 2. They are decreasing function (i.e., $\kappa(t) \to 0$) as the time goes infinity. In the case of an exponential relaxation (i), the Fano factor $F(t)$ becomes a double exponential function as shown in the table. In the case of inverse power function (ii), the Fano factor takes the form t^{ϵ} in the time region $t \to \infty$: (a) subdiffusion for $\epsilon = 2\alpha\kappa/\gamma < 1$ and (b) superdiffusion for $\epsilon = 2\alpha\kappa/\gamma > 1$. On the other hand, in the case of the power relaxation (iii), the Fano factor $F(t)$ becomes the fractional power exponential function of time. To understand the feature of temporal variation, numerical examples are depicted in Figures 2(a) and 2(b) as well as Figures 3(a) and 3(b).



(a) $\kappa(t)$

(b) $K(t) = \int_0^t \kappa(s)ds$

Figure 2: Time dependence of three relaxation functions for $\kappa(t)$ in Table 2: (i) an exponential function (solid line), (ii) an inverse power function (dotted line), and (iii) a fractional power function (dashed line); the values of parameters are $\kappa = 1$, $\gamma = 0.85$, $\alpha = 1$, and $\beta = 1$.

3.4. Nonstationary Yule-Furrey Process

When $\beta = 0$, $\lambda(0,t) = 0$ in (3.3). So one must omit (3.2) (i.e., one must redefine the range of variation) for the case of a generalized non-stationary Yule-Furrey process as follows:

$$\frac{d}{dt}p(n,t) = \lambda(n-1,t)p(n-1,t) - \lambda(n,t)p(n,t) \quad (n \geq 1). \tag{3.19}$$

The solution of $g(z,t)$ under the initial condition $p(n,0) = \delta_{n,1}$ becomes

$$\begin{aligned} g(z,t) &= \left(\frac{z\exp(-\Lambda(t))}{1 - z[1 - \exp(-\Lambda(t))]} \right) \\ &= [z\exp(-\Lambda(t))] \sum_{n=0}^{\infty} [1 - \exp(-\Lambda(t))]^n z^n. \end{aligned} \tag{3.20}$$

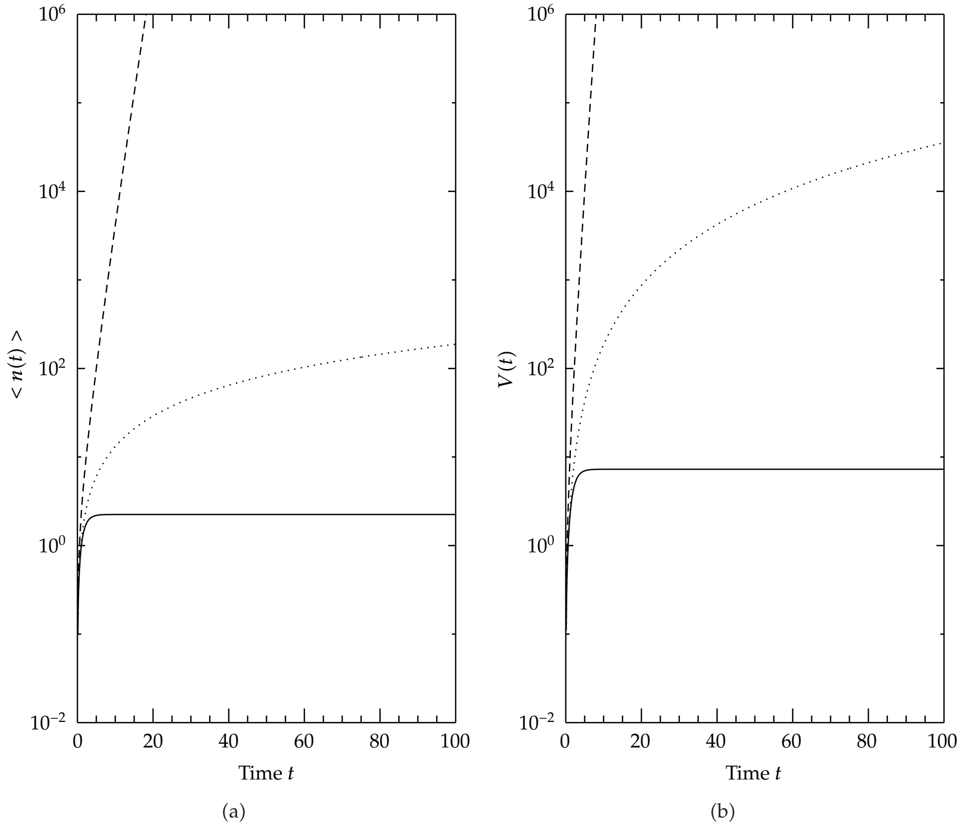


Figure 3: Time-dependence of the mean and the variance of three relaxation functions for $\kappa(t)$ in Table 2: (i) an exponential relaxation (solid line), (ii) an inverse power function (dotted line), (iii) a fractional power function (dashed line); the values of parameters are $\kappa = 1$, $\gamma = 0.85$, $\alpha = 1$, and $\beta = 1$.

The corresponding probability density $p(n,t)$ is obtained as

$$p(n,t) = \exp(-\Lambda(t))\bigl(1 - \exp(-\Lambda(t))\bigr)^{n-1}. \tag{3.21}$$

This is the geometric distribution

$$f(\theta) = \theta(1-\theta)^{n-1}, \tag{3.22}$$

with the parameter $\theta = \exp(-\Lambda(t))$, which is a special case of the Pascal (the negative binomial) distribution in (3.16). The mean and the variance are obtained as

$$\langle n(t)\rangle = \exp(\Lambda(t)), \qquad V(t) \equiv \sigma_n^2(t) = \exp(\Lambda(t))\bigl[\exp(\Lambda(t)) - 1\bigr]. \tag{3.23}$$

The Fano factor becomes

$$F = \frac{V(t)}{\langle n(t)\rangle} = \exp(\Lambda(t)) - 1 > 0. \tag{3.24}$$

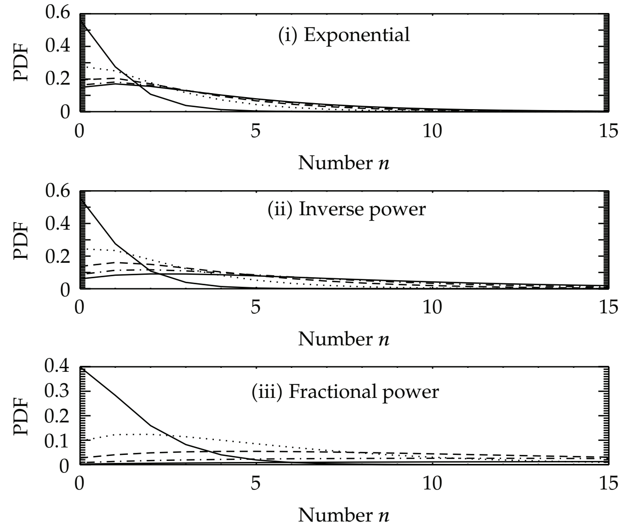


Figure 4: Time dependence of $p(n,t)$ in (3.15) for three relaxation functions for $\kappa(t)$ in Table 2: (i) an exponential function, and (ii) an inverse power function, (iii) a fractional power function; the values of parameters are $\alpha = 1$, $\kappa = 1$, and $\gamma = 0.85$. The pdf profiles are depicted for $t = 0.4$ (solid line), $t = 1.2$ (dotted line), $t = 2.0$ (dashed line), $t = 2.8$ (dash-dotted line) and $t = 3.6$ (dash-dotted line).

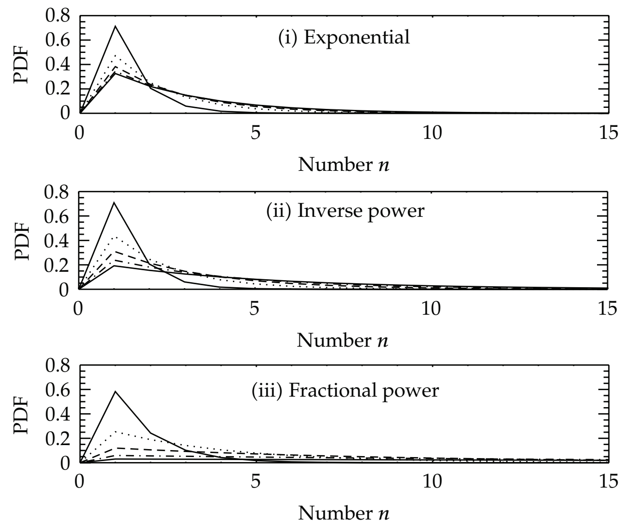


Figure 5: Time-dependence of $p(n,t)$ in (3.21) for three relaxation functions for $\kappa(t)$ in Table 2: (i) an exponential function, (ii) an inverse power function, (iii) a fractional power function; the values of parameters are $\alpha = 1$, $\kappa = 1$, and $\gamma = 0.85$. The pdf profiles are depicted for $t = 0.4$ (solid line), $t = 1.2$ (dotted line), $t = 2.0$ (dashed line), $t = 2.8$ (dash-dotted line), and $t = 3.6$ (dash-dotted line).

This means that the nature of statistics (Sub-Poissonian (SUBP, $F < 1$), Poissonian (P, $F = 1$), and Super-Poissonian (SUPP, $F > 1$)) changes depending on the functional form of $\lambda(t)$ and its parameter values involved. It is important to make notice of the fact that the variability of $F(t)$ changes in the two cases for $\beta \neq 0$ and $\beta = 0$. They are summarized in Table 3.

Table 3: Summary of the generalized Polya and non-stationary Yule-Furrey process.

Parameters	$P(\theta)$	$\langle n(t)\rangle$	$V(t)$	F
(i) $\alpha \neq 0, \beta \neq 0, n_0 = 0$	$\binom{n+r+1}{n}\theta^r(1-\theta)^n$	$r(\theta^{-1}-1)$	$r\theta^{-1}(\theta^{-1}-1)$	$F > 1$
(ii) $\alpha \neq 0, \beta = 0, n_0 = 1$	$\theta(1-\theta)^{n-1}$	θ^{-1}	$\theta^{-1}(\theta^{-1}-1)$	$F > 0$

In Table 3, θ, r, and F are defined by

$$\theta = \exp(-\Lambda(t)), \quad r = \frac{\beta}{\alpha}, \quad \Lambda(t) = \alpha\int_0^t \kappa(\tau)d\tau, \quad F = \frac{V(t)}{\langle n(t)\rangle}. \tag{3.25}$$

The temporal development of the probability density $p(n,t)$ for the generalized Polya process in (3.15) and the generalized Yule-Furry process in (3.21) for these three examples is depicted in Figures 4 and 5.

4. Discussions

4.1. Solvability Condition

We have studied the generalized Polya process with the transition rate in (3.3). How is the solvability if the transition rate is a more general one than that of (3.3) as

$$\lambda(n,t) = \alpha(t)n + \beta(t), \tag{4.1}$$

with $\alpha(t)$ and $\beta(t)$ being arbitrary functions with time, how is the solvability. In this case, the exact analytic solution is not obtained. The solvability condition is equivalent to the fact that $g(z,t)$ is written in the form of (3.5); $g(z,t) = C(z)G(z,t)$. For the transition rate in (4.1), the time-independent function $C(z)$ reduces to

$$C(z) = C_0 z^{-\beta(t)/\alpha(t)}. \tag{4.2}$$

This means that $\alpha(t)$ and $\beta(t)$ must have the same time-dependent scaling function $\kappa(t)$ with $\alpha(t) = \kappa(t)\alpha$ and $\beta(t) = \kappa(t)\beta$ (i.e., $\lambda(t) = \kappa(t)(\alpha n+\beta)$ in (3.3)) to get the exact analytic solution.

4.2. Master Equation in Memory Function Formalism

An alternative master equation in the memory function (MF) formalism for the generalized Polya process in (3.1) and (3.2) may be written as

$$\begin{gathered}\frac{d}{dt}p(n,t) = \int_0^t \phi(t-\tau)\left[\alpha(n-1)+\beta\right]p(n-1,\tau)d\tau - \int_0^t \phi(t-\tau)\left[\alpha n+\beta\right]p(n,\tau)d\tau \quad (n \geq 1),\\ \frac{d}{dt}p(0,t) = -\int_0^t \phi(t-\tau)\beta p(0,\tau)d\tau \quad (n = 0),\end{gathered} \tag{4.3}$$

where α and β are constants ($\alpha \neq 0$ and $\beta \neq 0$). The Laplace transform of the memory function is defined by $\phi[s] = \int_0^\infty \phi(t)\exp(-st)ds$. For $n \geq 1$, the recursion relation is obtained for the Laplace transform $p[n,s]$ of $p(n,t)$ as

$$\{s + (\alpha n + \beta)\phi[s]\}p[n,s] = \phi[s]\{\alpha(n-1) + \beta\}p[n-1,s] \quad (n \geq 1). \tag{4.4}$$

The general formal solution is given under the initial condition $p(n,0) = \delta_{n,0}$ in terms of the inverse Laplace transform as

$$p(n,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\prod_{k=0}^{n-1}\{(\alpha k + \beta)\phi[s]\}}{\prod_{k=0}^{n}\{s + (\alpha k + \beta)\phi[s]\}}\exp(st)ds. \tag{4.5}$$

When the memory function $\phi(t)$ or the pausing time distribution $\psi(t)$ is given (i.e., the Laplace transform of $\psi(t)$ is related to $\phi[s]$ as $\psi[s] = \phi[s]/(s + \phi[s])$), the probability density in (4.5) can be evaluated numerically. The explicit analytic expressions are obtained only for a few special cases [1] with $\alpha = 0$. The two formalisms have different features complement with each other (cf. Montroll and Shlesinger [1], Tokuyama and Mori [5], and Hänggi and Talkner [6]).

5. Concluding Remarks

In this paper, it is shown that there are two types of generalized master equation: (i) the memory function (MF) formalism in (1.1) and (ii) the convolution-less, non-stationary (NS) formalism in (1.2). Then, we propose a new model in the NS formalism: a generalized Polya process in (3.1) and (3.2) with the transition rate $\lambda(t) = \kappa(t)(\alpha n + \beta)$ having an arbitrary time-varying function $\kappa(t)$ in (3.3). Further, we exhibit the exact analytic solutions of the probability density $p(n,t)$ and the mean $\langle n(t)\rangle$ and variance $\sigma_n^2(t)$ for an arbitrary function $\kappa(t)$ of time. For some typical examples of $\kappa(t)$, the temporal variations of the mean and the variance are numerically exhibited.

There are many potential applications of the master equation in the NS formalism to non-equilibrium phenomena. In biological systems, the human EEG response to light flashes [12] (i.e., microscopic molecular transport associated with transient visual evoked potential (VEP)) and the transition phenomenon from spiral wave to spiral turbulence in human heart [13] can be formulated by the master equation in the NS formalism. In considering a stochastic model of infectious disease like a stochastic SIR model [14], the introduction of temporal variation of infection rate on account of various environmental changes leads to the master equation in the NS formalism.

In auditory-nerve spike trains, there are interesting observations [15, 16] that (i) the Fano factor $F(t)$ exhibits temporal variation $F(t) < 1$ in the intermediate time region and (ii) $F(t)$ also shows fractional power dependence t^ϵ in the time region $t \to \infty$ (ϵ = noninteger number). In the generalized Polya process, time variation of the Fano factor $F(t)$ (i.e., SUBP($F < 1$), P ($F = 1$), and SUPP ($F < 1$)) changes depending on the choice of the relaxation function $\kappa(t)$ and the values of α and β (cf. Tables 2 and 3). The related discussions in detail will be reported elsewhere.

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