Hindawi Publishing Corporation
Advances in High Energy Physics
Volume 2010, Article ID 427891, 30 pages
doi:10.1155/2010/427891

Review Article

The Master Space of Supersymmetric Gauge Theories

Amihay Hanany[1] and Alberto Zaffaroni[2, 3]

[1] *Theoretical Physics Group, Blackett Laboratory, Imperial College, London SW7 2AZ, UK*
[2] *Dipartimento di Fisica, Università di Milano-Bicocca, 20126 Milano, Italy*
[3] *INFN, Sezione di Milano-Bicocca, 20126 Milano, Italy*

Correspondence should be addressed to Alberto Zaffaroni, alberto.zaffaroni@mib.infn.it

Received 10 May 2010; Accepted 14 June 2010

Academic Editor: Carlos Nunez

We give a short review on the study of the moduli space and the spectrum of chiral operators for gauge theories living on branes at singularities. We focus on theories with four real supercharges in 3+1 and 2+1 dimensions. The theories are holographically dual to $AdS_5 \times H_5$ or $AdS_4 \times H_7$ backgrounds, in Type-IIB or -M theory, respectively. We demonstrate that most of the information on the moduli space and spectrum of the quiver gauge theories is encoded in the concept of the "Master Space", which is roughly the full moduli space for one brane, consisting of mesonic and baryonic degrees of freedom. We summarize the relevant information in generating functions for chiral operators, which can be computed using plethystics techniques and the language of complex geometry.

1. Introduction

We discuss the properties of a class of supersymmetric gauge theories that arise in brane constructions and in the AdS/CFT correspondence. We focus on the case of theories with four real supercharges. In this case the structure of the moduli space and the spectrum of BPS operators is conveniently described in the language of complex geometry and most of the results can be summarized in terms of supersymmetric generating functions for BPS states. The corresponding generating function reveals a rich structure of the chiral ring, its generators and their relations, and shares information about the dimension of the moduli space of vacua and the effective number of degrees of freedom in the system.

Generating functions for chiral operators in four-dimensional supersymmetric gauge theories have been extensively studied in the past years, ranging from SQCD [1–4] to quiver gauge theories living on branes at singularities [5–14]. The computation of such generating functions is a very hard problem but it can be simplified in particular circumstances. In the case of classical SQCD the simplification arises from the fact that there is no superpotential for the chiral fields. In the case of superconformal quiver gauge theories, one can exploit the fact that they have an $AdS_5 \times H_5$ or $AdS_4 \times H_7$ dual [15–17]. Information from the field theory and from the holographic dual can be combined to give a better understanding of the superconformal theory.

In this review article, we consider in detail the case of branes at CalabiYau singularities which are relevant for the AdS/CFT correspondence in three and four dimensions. A long standing problem is the reconstruction of the gauge theory from the dual geometry. This problem has been solved in four dimensions for toric CalabiYau using the language of Tilings [18, 19], providing the complete dictionary that relates toric CalabiYau to the corresponding quiver gauge theory. Some progress has been recently made also in three dimensions using Chern-Simons theories [20–27]. We demonstrate the importance of the concept of the master space, which is roughly the moduli space for one single brane, in all of these investigations.

The paper is organized as follows. In Section 2 we introduce the concept of generating functions and Hilbert series and provide some simple examples. In Section 3 we discuss the case of D3-branes at CalabiYau threefold singularities in Type-IIB. We introduce and study quivers for orbifold and conifold singularities and analyze the moduli space and the spectrum of chiral operators. We first discuss the case of one brane and introduce the concept of master space [14, 28], which is the space of solution of F-terms conditions. The master space contains information on both the mesonic and baryonic directions in the moduli space for one as well as for many branes. We then discuss the case of many branes and demonstrate how it is determined by the master space. In Section 4 we discuss the case of membranes at CalabiYau fourfold singularities in M theory. We demonstrate how the use of $\mathcal{N} = 2$ Chern-Simons theories helps in writing quiver theories with CalabiYau four moduli space and the role of the master space. We illustrate the general results with a choice of selected quivers which play an important role in both four- and three-dimensional gauge theories.

2. Generating Functions for Supersymmetric Gauge Theories

An important role in the study of supersymmetric gauge theories is played by chiral operators, which are annihilated by the supercharges of one chirality, for example $\overline{Q}_\alpha$. It is convenient to work modulo operators of the form $\{\overline{Q}_\alpha, \ldots\}$, which are called descendants and give a vanishing contribution to the correlation functions of chiral operators. Chiral operators have no short distance singularities in the Operator Product Expansion and therefore their product is well defined and it is still a chiral operator (see e.g., [29]). We then define the *chiral ring* as the set of chiral operators modulo descendants.

When we consider an $\mathcal{N} = 1$ supersymmetric gauge theory with vector multiplets W_α and a collection of charged chiral superfields Φ_i interacting with a superpotential $W(\Phi_i)$, the chiral operators are given by the lowest components of gauge-invariant products of Φ_i and W_α. The equation of motion

$$\overline{D}_\alpha \overline{D}^\alpha \overline{\Phi}_i = \partial_{\Phi_i} W(\Phi) \tag{2.1}$$

implies that all derivatives of the superpotential are descendants and vanish in the chiral ring. It is then very easy to write all chiral operators made with Φ_i. They are given by all possible gauge-invariant products of the scalar fields ϕ_i subject to the F-term constraint $\partial_{\phi_i} W(\phi) = 0$, where ϕ_i is the lowest component of the superfield Φ_i.

Insertions of W_α give more general scalar and fermionic chiral operators. However, since the maximum number of insertion of W_α preserving chirality is two, the set of chiral operators involving W_α can be reconstructed from those made with Φ_i. We refer to [30] for a general discussion of the role of W_α and to [13] for the case of superconformal quivers.

Given a $\mathcal{N} = 1$ supersymmetric gauge theory with a collection of $U(1)$ global symmetries $\prod_{i=1}^{G} U(1)_i$, we can introduce a set of auxiliary parameters (fugacities) $\{t_i\}_{i=1}^{G}$ and define the generating function for the chiral ring

$$g(\{t_i\}) = \sum_{k_1,\ldots,k_G} n_{k_1,\ldots,k_G} t_1^{k_1} \cdots t_G^{k_G}, \tag{2.2}$$

where $n_{k_1,\ldots,k_G}$ is the number of operators with charges $\{k_1,\ldots,k_G\}$ under the global $U(1)^G$ symmetry. If part of the global symmetry is nonAbelian, one can introduce a fugacity per each $U(1)$ symmetry in the Cartan subalgebra of this nonAbelian group. The number of chiral operators is typically infinite, but, if we consider enough global symmetries, the numbers $n_{k_1,\ldots,k_G}$ become nonnegative integers. In mathematical language, we compute the *Hilbert series* of a graded ring.

In the superconformal case we give an alternative interpretation to the Hilbert series. The $\mathcal{N} = 1$ superconformal algebra always contains a generator for the R symmetry denoted by $U(1)_R$. Chiral operators are the lowest components of a short (BPS) multiplet and are protected by supersymmetry and by the conformal algebra. In particular, their dimension Δ and R charge are related by $\Delta = 3R/2$. By restricting to just one parameter t for the R symmetry we can write a single variable generating function (which is termed unrefined Hilbert series)

$$g(t) = \sum_k n_k t^k \tag{2.3}$$

counting the number n_k of chiral operators of R-charge equal to k. Given the relation $\Delta = 3R/2$, we are equivalently counting chiral operators with a given dimension.

From a statistical mechanics perspective the functions $g(\{t_i\})$ and $g(t)$ are interpreted as partition functions that count chiral operators. For each conserved charge k_i one introduces a conjugate parameter μ_i called the chemical potential and fugacity t_i related to the chemical potential by $t_i = e^{-\mu_i}$. With this interpretation the function $g(\{t\})$ is the partition function in the grand-canonical ensemble for which μ is fixed and n_k is the partition function in the canonical ensemble in which the charge k is fixed. The transition between $g(t)$ and n_k is given by the discrete Legendre transform (2.3) and the inverse is given by a contour integral

$$n_k = \oint_{|t|=1} \frac{dt}{2\pi i t} t^{-k} g(t). \tag{2.4}$$

We can take a complementary point of view and consider the chiral operators as functions on the moduli space of the supersymmetric gauge theory. The vacuum moduli space, $\mathcal{M}$, is given by the vanishing of the scalar potential as a function of the scalar components ϕ_i of the superfields of the field theory. This, in turn, is the set of zeros of D-terms and F-terms modulo gauge transformations. It is a standard fact that the D-term conditions and the gauge transformations can be exchanged by modding out by the complexified gauge group. In a mathematical language, $\mathcal{M}$ is the symplectic quotient of the space of F-term solutions by the action of the complexified gauge symmetries. We denote the space of solutions of F-terms by $\mathcal{F}^{\flat}$ and the symmetries by $G_{D^{\flat}}$, then we have

$$\mathcal{M} \simeq \mathcal{F}^{\flat} // G_{D^{\flat}}. \tag{2.5}$$

Chiral operators are invariant under the complexified gauge group, functions of ϕ_i and no $\overline{\phi}$, and are subject to the F-terms. They are therefore holomorphic functions on the moduli space. Vice versa, we can associate a chiral operator to any *holomorphic function* on the moduli space. We can then reinterpret (2.2) as counting holomorphic functions on the moduli space.

It is a general mathematical fact that $g(t)$ is a rational function which can be written as

$$g(\mathrm{t}) = \frac{P(t)}{Q(t)} \tag{2.6}$$

for some polynomials $P(t)$ and $Q(t)$. It is a well-known and interesting fact that $g(t)$ has a pole at $t \to 1$ of order equal to the dimension of the moduli space. $Q(t)$ encodes the generators of the moduli space and can always be written in a factorized form

$$Q(t) = \prod_i \left(1 - t^{n_i}\right), \tag{2.7}$$

where the product runs over all generators and n_i is the charge of the ith generator. $P(t)$ encodes the relations satisfied by the generators. In very special cases $P(t)$ can be written in a factorized form, and then $g(t)$ is said to be of an Euler form,

$$P(t) = \prod_j \left(1 - t^{m_j}\right), \tag{2.8}$$

where the product runs over all possible relations and m_j is the degree of the jth relation. In such cases the moduli space is said to be a complete intersection and enjoys the special property that the number of relations plus the dimension of the moduli space is equal to the number of generators.

If the theory has a nonAbelian global symmetry, G, then it is possible to write the function $g(\{t_i\})$ as a sum over characters of irreducible representations of G:

$$g(\{t_i\}) = \sum_R n_R \chi_R(\{t_i\}), \tag{2.9}$$

where χ_R is the character of the irreducible representation R and n_R is the number of times this representation appears on the moduli space.

The structure of the moduli space of a supersymmetric gauge theory is usually complicated, and should be best cast in the language of algebraic varieties. Typically, $\mathcal{M}$ consists of a union of various branches, such as the mesonic branch or the baryonic branch, the Coulomb branch or the Higgs branch; the names are chosen according to some characteristic property of the specific branch.

Correspondingly, the generating function (2.2) is hard to compute, even at the classical level. In principle, thanks to the results of commutative algebra, generating functions like (2.2) can be computed in an algorithmic way. We first compute the Hilbert series of an extended ring consisting of all the elementary fields modded out by the ideal of F-terms and graded with the parameter t_i and an extra set of gauge charges z_k labeling the Cartan part of the gauge group. This computation is algorithmic and can be performed with computer algebra programs like Macaulay2 [31]. We then project on the set of gauge-invariants by averaging on the parameters z_i with the Haar-Weyl measure of the gauge group. We refer to [12] for a discussion and examples. This procedure works well in the case where there are no F-terms (e.g., SQCD, see [1, 3, 32]) and for theories with small number of fields and moderate values of N.

Let us just examine an explicit example. To study it properly we need the following definition.

Notations. the *plethystic exponential* of a multivariable function $f(t_1,\ldots,t_n)$ that vanishes at the origin, $f(0,\ldots,0)=0$, is defined to be

$$\mathrm{PE}\left[f(t_1,\ldots,t_n)\right] := \exp\left(\sum_{r=1}^{\infty}\frac{f(t_1^r,\ldots,t_n^r)}{r}\right). \tag{2.10}$$

The *fermionic plethystic exponential* of a multivariable function $f(t_1,\ldots,t_n)$ that vanishes at the origin, $f(0,\ldots,0)=0$, is defined to be

$$\mathrm{PE}_F\left[f(t_1,\ldots,t_n)\right] := \exp\left(\sum_{r=1}^{\infty}\frac{(-1)^{r+1}f(t_1^r,\ldots,t_n^r)}{r}\right). \tag{2.11}$$

As discussed in detail below, the plethystic exponential is the right tool to study symmetric products. In the case of a moduli space with a single generator with fugacity t and no relations, we can write

$$g(t) = \mathrm{PE}[t] = \frac{1}{1-t} \tag{2.12}$$

More generally, the plethystic exponential of $g(t_1,\ldots,t_n) = t_1+\cdots+t_n$ gives the Hilbert series for a freely generated moduli space with n generators with fugacities $t_1,\ldots,t_n$:

$$\mathrm{PE}[t_1+\cdots+t_n] = \frac{1}{(1-t_1)\cdots(1-t_n)}. \tag{2.13}$$

Similarly the fermionic analog is

$$\mathrm{PE}_F[t_1+\cdots+t_n] = (1+t_1)\cdots(1+t_n). \tag{2.14}$$

2.1. A Simple Example

As a simple example, let us consider the gauge theory which consists of an $SU(N)$ gauge group with matter given by 1 chiral multiplet transforming in the adjoint representation, 1 transforming in the fundamental representation, and 1 in the antifundamental representation, with $W = 0$. Details of this theory and the computations can be found in [32]. There are three $U(1)$ global symmetries that act on each of these representations. Let us denote the corresponding fugacities by $t_a, t_f, t_{\tilde{f}}$, respectively. Since $W = 0$, the F-flat moduli space is simply given by $\mathbb{C}^{N^2-1+2N}$. As an algebraic variety, it is freely generated by the $N^2 - 1 + 2N$ elementary fields. The generating function takes the form of a rational function as in formula (2.6):

$$g_{\mathcal{F}^\flat}\left(t_a, t_f, t_{\tilde{f}}\right) = \frac{1}{(1-t_a)^{N^2-1}(1-t_f)^N\left(1-t_{\tilde{f}}\right)^N}. \tag{2.15}$$

The trivial numerator reflects the absence of relations among the generators, and the denominator has the form given in (2.7), where the product runs over the elementary fields.

To obtain the partition function for the theory, we need to project on gauge-invariant operators. We first introduce fugacities z_i for the Cartan subalgebra of $SU(N)$ and write a refined generating function

$$g_{\mathcal{F}^\flat}\left(t_a, t_f, t_{\tilde{f}}; z_1, \ldots, z_{N-1}\right) = \mathrm{PE}\Big[[1,0,\ldots,0,1]t_a + [1,0,\ldots,0]t_f + [0,\ldots,0,1]t_{\tilde{f}}\Big], \tag{2.16}$$

where $[n_1, \ldots, n_{N-1}]$ denotes the character for the irreducible representation of $SU(N)$ with Dynkin labels $n_1, \ldots, n_{N-1}$. The projection on gauge-invariants can be done by integrating over the Haar measure of $SU(N)$,

$$\int \mu(\{z_i\}) g_{\mathcal{F}^\flat}\left(t_a, t_f, t_{\tilde{f}}; z_1, \ldots, z_{N-1}\right). \tag{2.17}$$

Explicit computation can be made for small values of N and the final result can be obtained by observing that the moduli space is a complete intersection. It is easy to find the generators of the moduli space. Denote the lowest components of the 3 chiral multiplets by $\phi, Q, \overline{Q}$. Then the generators are

$$M_k = \overline{Q}\phi^k Q, \quad k = 0, \ldots, N-1, \qquad u_k = \frac{1}{k}\mathrm{Tr}\left(\phi^k\right), \quad k = 2, \ldots, N \tag{2.18}$$

giving N adjoint mesons and $N-1$ adjoint invariants (operators with higher values of k are not independent as discussed below), together with 2 adjoint baryons

$$B = \epsilon^{i_0 i_1 \cdots i_{N-1}} Q_{i_0} (\phi Q)_{i_1} \cdots \left(\phi^{N-1} Q\right)_{i_{N-1}}, \qquad \overline{B} = \epsilon_{i_0 i_1 \cdots i_{N-1}} \overline{Q}^{i_0} \left(\overline{Q}\phi\right)^{i_1} \cdots \left(\overline{Q}\phi^{N-1}\right)^{i_{N-1}}. \tag{2.19}$$

These are $2N + 1$ generators and since on a generic point on the moduli space the gauge group is completely Higgsed, the dimension of the moduli space is $2N$, meaning that there is one relation which is satisfied between these generators. This relation can be derived by constructing the $N \times N$ symmetric matrix A with entries $A_{i,j} = M_{i+j-2}$. The entries of this matrix are the gauge-invariant adjoint mesons $M_k, k = 0, \ldots, 2N-2$, and it is easy to see that the determinant satisfies

$$\det(A) = \overline{B}B. \tag{2.20}$$

The higher-order invariants can be expressed in terms of lower-order invariants by a use of the Cayley-Hamilton theorem which states that the characteristic polynomial of the matrix ϕ

$$P(x) = \det(x - \phi) = \sum_{j=0}^{N} s_{N-j} x^j, \quad s_0 = 1,\ s_1 = 0, \tag{2.21}$$

satisfies the relation $P(\phi) = 0$ as a matrix relation, and hence multiplying by $\overline{Q}$ to the left and by $\phi^k Q$ to the right one gets,

$$\sum_{j=0}^{N} s_{N-j} M_{j+k} = 0, \quad k = 0, \ldots, N-2, \tag{2.22}$$

giving $N-1$ relations among $(2N-1)+(N-1) = 3N-2$ variables, M_k and s_j. These relations can be used in order to express M_k for the values $k = N, \ldots, 2N-2$ in terms of M_k $k = 0 \ldots, N-1$ and the s_j. The relations between the variables s_j and u_j are well known and go under the name Newton relations,

$$n s_n + \sum_{j=1}^{n} j u_j s_{n-j} = 0, \quad n = 2, \ldots, N. \tag{2.23}$$

One can also view this as a complete intersection moduli space which is defined over the $(2N-1)+2+(N-1)+(N-1) = 4N-1$ variables $M_k, \overline{B}, B, u_k, s_k$ which are subject to the $1+(N-1)+(N-1) = 2N-1$ relations (2.20), (2.22) and (2.23) thus giving a $2N$ dimensional moduli space.

To compute the Hilbert series, we first need the 3 fugacities for each gauge-invariant under the global symmetry $U(1)_f \times U(1)_a \times U(1)_{\overline{f}}$. These are counting the number of $\overline{Q}$'s, ϕ's, and Q's in each gauge-invariant: $t_{\overline{f}} t_a^k t_f$ for M_k, $t_{\overline{f}}^N t_a^{N(N-1)/2}$ for $\overline{B}$, and $t_a^{N(N-1)/2} t_f^N$ for B. The resulting Hilbert series is given by writing the generators in the denominator and the relations in the numerator

$$g\left(t_{\overline{f}}, t_a, t_f\right) = \frac{1 - t_{\overline{f}}^N t_a^{N(N-1)} t_f^N}{\left(1 - t_{\overline{f}}^N t_a^{N(N-1)/2}\right)\left(1 - t_a^{N(N-1)/2} t_f^N\right)\left(\prod_{k=0}^{N-1}\left(1 - t_{\overline{f}} t_a^k t_f\right)\right)\left(\prod_{k=2}^{N}\left(1 - t_a^k\right)\right)}. \tag{2.24}$$

Note that such a simple form is only possible when the moduli space is a complete intersection. In more general cases, other methods for computing the Hilbert series need to be applied, as outlined above.

3. D3-Branes at Singularities

Closely related to the AdS/CFT correspondence is the topic of D3-branes living at conical singularities. In this context the moduli space $\mathcal{M}$ has an elegant geometrical realization. When D3-branes are transverse to an affine (noncompact) threefold CalabiYau cone $\mathcal{X}$, a supersymmetric gauge theory exists on the world-volume of the branes. This is a gauge theory of quiver type with $U(N)$ gauge groups and bifundamental or adjoint fields. By taking the near horizon limit of the geometry induced by the D3-branes we obtain the string background $AdS_5 \times H_5$, where H_5 is the Sasaki-Einstein base of the cone $\mathcal{X}$. It is then a prediction of the AdS/CFT correspondence that the theory on N physical D3-branes at the singularity flows in the IR to a superconformal field theory [15, 17]. The isometries of the (noncompact) CalabiYau becomes global symmetries in the dual theory, which are usually referred as *mesonic*. The maximum rank of the mesonic symmetry group is three, which is realized for toric CalabiYau singularities. Our main interest is the IR physics of this system, where conformal invariance holds; in this limit all Abelian gauge symmetries become weakly coupled and remain as global symmetries called *baryonic* symmetries. The remaining gauge symmetry $G^{\text{n.a.}}_{D^\flat}$ is fully nonAbelian, typically given by products of $SU(N)$ groups. It is then natural to distinguish between mesonic and baryonic operators. A mesonic operator has zero baryonic charge, and thus it is invariant under the Abelian $U(1)$ factors.

Under these circumstances, the moduli space, obtained by dividing by the non-Abelian gauge group,

$$\mathcal{M} \simeq \mathcal{F}^\flat // G^{\text{n.a.}}_{D^\flat} \tag{3.1}$$

is a combined mesonic and baryonic moduli space. These mesonic and baryonic branches are not necessarily separate (irreducible) components of $\mathcal{M}$ but are instead in most cases intrinsically merged into one or more components in $\mathcal{M}$. Even when mesonic and baryonic directions are mixed, it still makes sense to talk about the more familiar mesonic moduli space ${}^{\text{mes}}\mathcal{M}$, as the subvariety of $\mathcal{M}$ parameterized by mesonic operators only. The mesonic moduli space can be obtained as a further quotient of $\mathcal{M}$ by the Abelian symmetries:

$${}^{\text{mes}}\mathcal{M} \simeq \mathcal{M} // U(1)_{D^\flat}. \tag{3.2}$$

It is of particular interest to consider the case of a single D3-brane transverse to the CalabiYau threefold $\mathcal{X}$, which enlightens the geometrical interpretation of the moduli space. The gauge theory on a single D3-brane is an Abelian $U(1)^g$ theory. Since the motion of the D-brane is parameterized by this transverse space, the moduli space of this Abelian theory must coincide with the noncompact CalabiYau threefold $\mathcal{X}$ transverse to the D3-brane and has complex dimension three. However, we could examine the IR limit of the theory where the entire gauge group decouples. In this situation, the moduli space $\mathcal{M}$ is given by the space of F-flatness $\mathcal{F}^\flat$. $\mathcal{M}$ is an algebraic variety of dimension greater than three, which contains properly $\mathcal{X}$ as a subvariety. We could identify $\mathcal{X}$ as the mesonic part of the moduli space.

Geometrically, $\mathcal{F}^b$ is a fibration over the mesonic moduli space $\mathcal{X}$ given by relaxing the $U(1)$ D-term constraints in (3.2). Physically, $\mathcal{F}^b$ is obtained by adding *baryonic* directions to the mesonic moduli space (Of course, we cannot talk about baryons for $N = 1$, but we can alternatively interpret these directions as Fayet-Iliopoulos (FI) parameters in the stringy realization of the $N = 1$ gauge theory. Indeed on the world-volume of a single D-brane there is a collection of $U(1)$ gauge groups, each giving rise to an FI parameter, which relax the D-term constraints.).

For $N > 1$, the situation is more subtle. Again, the common lore says that the moduli space is probed by a collection of N physical branes which are mutually BPS branes and thus can be arbitrarily located in the internal manifold. Hence the moduli space is given by the symmetrized product of N copies of $\mathcal{X}$. Actually this is only true for the mesonic moduli space. The full moduli space $\mathcal{M}$ is a bigger algebraic variety of more difficult characterization. In the next section this situation is elucidated and it is shown how the properties of $\mathcal{M}$ for arbitrary number of branes are encoded in the moduli space for a single brane. In view of the importance of the moduli space for one brane even for larger N, we adopt the important convention that the $N = 1$ moduli space is dubbed the *master space*.

Let us clarify the previously abstruse discussion with explicit examples.

3.1. The Case of One Brane: The Master Space

The moduli space for one brane is the space of solutions $\mathcal{F}^b$ of the F-terms where the fields are taken to be c-numbers. The generating function for one brane is just the Hilbert series of the quotient ring

$$\frac{\mathbb{C}[\phi_1, \ldots, \phi_E]}{\{dW = 0\}} \tag{3.3}$$

generated by the elementary fields ϕ_i modulo the ideal of F-terms. This generating function can be easily computed for all reasonable (= not too large) quivers by using computer algebra programs. Let us discuss some simple examples.

3.1.1. The Case of $\mathbb{C}^3$

The case of one brane probing $\mathcal{X} = \mathbb{C}^3$ is described by the $\mathcal{N} = 4$ SYM with gauge group $U(1)$. In $\mathcal{N} = 1$ notations, the theory has three chiral adjoint supermultiplets Φ_i interacting with the superpotential

$$W = \Phi_1[\Phi_2, \Phi_3]. \tag{3.4}$$

In the case of one brane, the lowest components ϕ_i are c-numbers and the F-terms are trivial. The master space is just described by three free complex variables ϕ_i. Since the $U(1)$ acts trivially on the adjoint fields, there is no distinction between the master space and the mesonic moduli space. Both coincide with $\mathbb{C}^3$. The gauge-invariant chiral operators are given by all the possible products $\phi_1^{n_1}, \phi_2^{n_2}, \phi_3^{n_3}$.

$\mathcal{X} = \mathbb{C}^3 = \mathbb{R}^6$ has a large isometry $SO(6)$, with rank three. The subgroup $U(3)$ acts in the obvious way on the three variables ϕ_i. The Cartan subgroup is $U(1)^3$, where the ith $U(1)$

acts on the corresponding variable $\phi_i \to e^{i\alpha}\phi_i$ for $i = 1, 2, 3$. We can introduce fugacities t_i for the three $U(1)$ actions and use them to grade the chiral operators. The resulting Hilbert series is

$$g_1\left(t_1, t_2, t_3, \mathcal{F}^{\flat}_{\mathbb{C}^3}\right) = \frac{1}{(1-t_1)(1-t_2)(1-t_3)}. \tag{3.5}$$

We can now introduce $SU(3)$ weights x_1, x_2 which reflect the fact that the chemical potentials t_1, t_2, t_3 are in the fundamental representation of $SU(3)$, and a chemical potential t for the $U(1)_R$ charge,

$$(t_1, t_2, t_3) = t\left(x_1, \frac{x_2}{x_1}, \frac{1}{x_2}\right). \tag{3.6}$$

We can get an expansion in terms of irreducible representations of $SU(3)$

$$g_1\left(t_1, t_2, t_3; \mathcal{F}^{\flat}_{\mathbb{C}^3}\right) = \mathrm{PE}[t[1,0]] = \sum_{n=0}^{\infty}[n,0]t^n. \tag{3.7}$$

where the symbol $[n, m]$ denotes the character of the $SU(3)$ representation with Dynkin labels (n, m). For example, for the fundamental representation we have

$$[1,0] = x_1 + \frac{x_2}{x_1} + \frac{1}{x_2}. \tag{3.8}$$

3.1.2. The Case of the Conifold

One of the most familiar examples of a conical CalabiYau singularity is the conifold $\mathcal{X} = \mathcal{C}$, described by the equation

$$xy = zw \tag{3.9}$$

in $\mathbb{C}^4$. This variety is a complete intersection in $\mathbb{C}^4$. In general the set of n algebraic equations in $\mathbb{C}^m$ is a complete intersection if it defines a variety of dimension $m - n$, meaning that each single equation reduces the complex dimension by exactly one unit. For the conifold, the dimension (3) can be obtained as the difference between the numbers of variables (4) and the number of equations (1).

The quiver is given in Figure 1. The gauge group is $U(1) \times U(1)$ and we have two bifundamental fields A_i with charges $(1, -1)$ and two fields B_i with charges $(-1, 1)$. The gauge theory has an explicit global symmetry $SU(2)_1 \times SU(2)_2 \times U(1)_R \times U(1)_B$ and the four fields transform under these symmetries according to Table 1. For one brane, the F-terms are trivial and the master space for the conifold is simply $\mathcal{F}^{\flat} = \mathbb{C}^4$, parameterized by four free variables

Table 1: The transformation, under the explicit global symmetry group $SU(2)_1 \times SU(2)_2 \times U(1)_R \times U(1)_B$, of the 4 fields in the conifold theory. The monomials indicate the associated chemical potentials in the Plethystic programme.

	$SU(2)_1$ $(2j_1, 2m_1)$	$SU(2)_2$ $(2j_2, 2m_2)$	$U(1)_R$	$U(1)_B$	Monomial
A_1	$(1, +1)$	$(0, 0)$	$\frac{1}{2}$	1	$t_1 x$
A_2	$(1, -1)$	$(0, 0)$	$\frac{1}{2}$	1	$\frac{t_1}{x}$
B_1	$(0, 0)$	$(1, +1)$	$\frac{1}{2}$	-1	$t_2 y$
B_2	$(0, 0)$	$(1, -1)$	$\frac{1}{2}$	-1	$\frac{t_2}{y}$

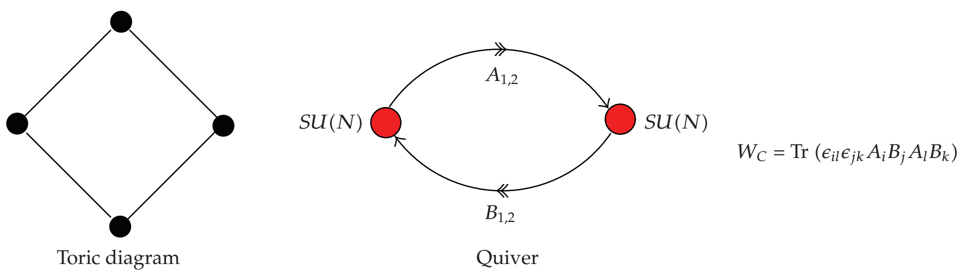

Figure 1: The quiver and toric diagrams, as well as the superpotential for the conifold $\mathcal{C}$.

$A_{1,2}$ and $B_{1,2}$ [8, 11]. We can introduce four fugacities for the four $U(1)$s in the symmetry group and write the refined Hilbert series for the master space $\mathcal{F}^{\flat}_{\mathcal{C}}$:

$$g_1\left(t_1, t_2, x, y; \mathcal{F}^{\flat}_{\mathcal{C}} = \mathbb{C}^4\right) = \frac{1}{(1 - t_1 x)(1 - t_1/x)(1 - t_2 y)(1 - t_2/y)}. \tag{3.10}$$

We used the choice of fugacities indicated in Table 1. x and y are fugacities for the Cartan subgroup of the two $SU(2)$ symmetries. t_1 and t_2, respectively, count the number of A and B fields. We could alternatively define b which counts baryon number and t which counts (twice) the total R-charge; then $t_1 = tb$ and $(t_2 = t/b)$. Indeed, if we remove the refinement by setting $t_1 = t,\ t_2 = t,\ x = 1,\ y = 1$, we would obtain the familiar Hilbert series for $\mathbb{C}^4$ which is $g_1(t; \mathbb{C}^4) = (1 - t)^{-4}$.

The Mesonic Moduli Space

The mesonic moduli space is obtained by further dividing by the gauge group $U(1) \times U(1)$. The overall $U(1)$ acts trivially and we are left with a $U(1)$ action where the A_i has charge $+1$ and the B_i charge -1. The mesonic gauge-invariant operators are

$$x = A_1 B_1, \quad y = A_2 B_2, \quad z = A_1 B_2, \quad w = A_2 B_1 \tag{3.11}$$

which satisfies (3.9) and correctly reproduce $\mathcal{C}$ as the mesonic moduli space.

We can easily write the refined Hilbert series for the mesonic moduli space using the fact that the manifold is a complete intersection

$$g_1\left(t, x, y;\ {}^{\mathrm{mes}}\mathcal{M} = \mathcal{C}\right) = \frac{1 - t^4}{\left(1 - t^2 xy\right)\left(1 - t^2 x/y\right)\left(1 - t^2 y/x\right)\left(1 - t^2/xy\right)}. \tag{3.12}$$

In this formula each factor in the denominator corresponds to a generator x, y, z, w. The numerator is associated with the relation $xy = zw$ (with fugacity t^4). The refined Hilbert series only depends on three fugacities t, x, y. In fact, the baryonic symmetry associated with b is acting trivially on the mesonic moduli space. There are exactly three remaining mesonic symmetries.

The refined Hilbert series can be easily expanded in terms of representation of the global symmetry. Denoting by $[n_1; n_2]$ the character for the representation of highest weight $n_1 = 2j_1$ and $n_2 = 2j_2$ of the symmetry group $SU(1)_1 \times SU(2)_2$ we have

$$g_1\left(t, x, y;\ {}^{\mathrm{mes}}\mathcal{M} = \mathcal{C}\right) = \sum_{n=0}^{\infty} [n; n] t^{2n}. \tag{3.13}$$

The t^{2n} terms in this sum are associated with the gauge-invariant operator $A_{i_1} B_{j_1} \cdots A_{i_n} B_{j_n}$ which has the correct R-charge and it is completely symmetric in the indices of the A's and in the indices of the B's, thus transforming in the $[n; n]$ representation of $SU(1)_1 \times SU(2)_2$.

We can see the master space as a $\mathbb{C}^*$ fibration over the mesonic moduli space obtained by relaxing the D-term constraint. In the case of the conifold $\mathcal{C}$, we divide the master space $\mathbb{C}^4$ by a $\mathbb{C}^*$ action corresponding to the $U(1)$ gauge group:

$$\mathcal{C} = \frac{\mathbb{C}[A_1, A_2, B_1, B_2]}{\{+1, +1, -1, -1\}}. \tag{3.14}$$

The $U(1)$ group corresponds to a baryonic symmetry, and we may as well say that we mod the master space by all the complexified baryonic symmetries.

This symplectic quotient description allows for an alternative derivation of the Hilbert series of $\mathcal{C}$ from the Hilbert series of the master space $\mathbb{C}^4$:

$$g_1\left(t, x, y;\ {}^{\mathrm{mes}}\mathcal{M} = \mathcal{C}\right) = \oint_{|b|=1} \frac{db}{2\pi i b} \frac{1}{(1 - tbx)(1 - tb/x)\left(1 - ty/b\right)\left(1 - t/by\right)}. \tag{3.15}$$

The integrand is just the fully refined Hilbert series for the master space as given in (3.10). By integrating over the fugacity b, which is associated with the $U(1)$ gauge action or, equivalently, with the baryonic number, we project on operators invariant under the $\mathbb{C}^*$ action. The contour integral should be done on the unit circle in the complex plane and takes contributions from the two poles inside the circle, $b = ty$ and $b = t/y$, where we have taken $|t| < 1$ in order to have a convergent geometrical series. An easy computation reproduces (3.12).

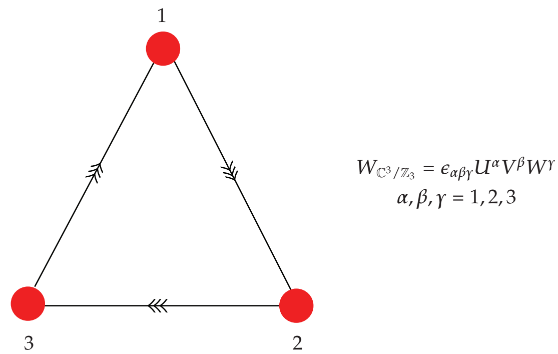

Figure 2: The quiver diagram and superpotential for dP_0.

3.1.3. The Case of $\mathbb{C}^3/\mathbb{Z}_3$

Let us next study the orbifold $\mathbb{C}^3/\mathbb{Z}_3$ with action $(1, 1, 1)$ on the $\mathbb{C}^3$. This is also referred to in the literature as dP_0, the cone over the zeroth del Pezzo surface. The quiver theory is summarized in Figure 2. For this case, the nine F-terms are

$$U^\alpha V^\beta = U^\beta V^\alpha, \quad V^\alpha W^\beta = V^\beta W^\alpha, \quad W^\alpha U^\beta = W^\beta U^\alpha, \quad \alpha \neq \beta. \tag{3.16}$$

In principle, nine equations for nine fields should reduce the moduli space to a point. However the variety (3.16) in $\mathbb{C}^9$ is an example of a set of equations which are not a complete intersection. It is easy to see that the generic point can be parametrized by five complex variables and therefore the manifold has dimension five.

There are various methods to compute the refined Hilbert series for the master space. The computation of the Hilbert series of quotient rings in commutative algebra is algorithmic and can be performed with computer algebra programs like Macaulay2 [31]. In the specific case, we can also follow an analytical route which is available for all toric quivers. The formulation in terms of Tilings [18, 19] allows to express all F-term constraints in terms of a Gauged Linear Sigma Model (GLSM) and reexpress the master space as a symplectic quotient. In the case of $\mathbb{C}^3/\mathbb{Z}_3$ this is done as follows.

The F-term constraints can be explicitly solved by introducing six auxiliary variables

$$U_i \longrightarrow p_i q_1, \quad V_i \longrightarrow p_i q_2, \quad W_i \longrightarrow p_i q_3. \tag{3.17}$$

In the Tiling description these arise as perfect matchings as shown in Figure 3. The previous parameterization is defined modulo a $\mathbb{C}^*$ action on the p_i with charge -1 and on the q_i with charge $+1$. We can therefore see the master space as the quotient

$$\mathcal{F}^\flat_{\mathbb{C}^3/\mathbb{Z}_3} = \frac{\mathbb{C}^6}{\{-1, -1, -1, 1, 1, 1\}}, \tag{3.18}$$

where we order the perfect matchings as $p_1, p_2, p_3, q_1, q_2, q_3$. We see from this description that the master space is a five-dimensional toric variety with $SU(3) \times SU(3) \times U(1)$ symmetry, where the first $SU(3)$ is visible in the quiver and superpotential as the global $SU(3)$ symmetry

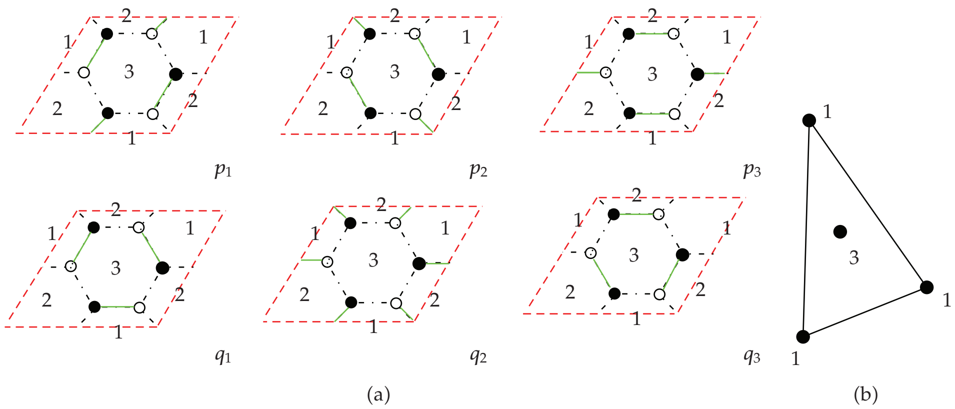

Figure 3: (a) The perfect matchings for the dimer model corresponding to dP_0, with the external perfect matchings p_i and the internal perfect matchings q_i; (b) The toric diagram of dP_0 with the labeled multiplicity of GLSM fields. The master space is given by the perfect matchings modulo relations [14]. There is one relation among perfect matching $p_1 + p_2 + p_3 = q_1 + q_2 + q_3$ and this gives the description of the master space as $\mathbb{C}^6/\{-1,-1,-1,1,1,1\}$. We refer to [18, 19] for details on the Tiling construction and to [14] for a detailed discussion of this example.

and the second $SU(3)$ is hidden. We can introduce weights for the action of the global symmetry on perfect matchings as follows:

$$p_1, p_2, p_3, q_1, q_2, q_3 \longrightarrow tx_1, \frac{t}{x_2}, \frac{tx_2}{x_1}, \frac{1}{y_1}, y_2, \frac{y_1}{y_2}, \tag{3.19}$$

where t is the $U(1)$ charge, x_1, x_2 are weights for the first $SU(3)$ and y_1, y_2 are weights for the second $SU(3)$. Note that these weights and the labels are chosen such that they indicate the highest weight states of the fundamental and antifundamental representations of $SU(3)$. Such weight assignments allows for an easier tracking of highest weight states for higher dimensional representations, when a character expansion is performed. The Hilbert series for the master space is then given by the Molien-Weyl formula

$$\begin{aligned} &g_1\left(t, x_1, x_2, y_1, y_2; \mathcal{F}^\flat_{\mathbb{C}^3/\mathbb{Z}_3}\right) \\ &\quad = \oint_{|z|=1} \frac{dz}{2\pi i z} \frac{1}{(1 - tx_1/z)(1 - tx_2/x_1 z)(1 - t/x_2 z)(1 - y_2 z)(1 - zy_1/y_2)(1 - z/y_1)}. \end{aligned} \tag{3.20}$$

The Hilbert series for just one charge is computed by setting the nonAbelian fugacities to 1

$$g_1\left(t; \mathcal{F}^\flat_{\mathbb{C}^3/\mathbb{Z}_3}\right) = \oint_{|z|=1} \frac{dz}{2\pi i z (1 - t/z)^3 (1 - z)^3} = \frac{1 + 4t + t^2}{(1 - t)^5}. \tag{3.21}$$

This expression has a pole of order five as expected for a five-dimensional variety. Taking the plethystic logarithm [7] of this expression we find 9 generators at order t subject to 9 relations at order t^2,

$$\mathrm{PE}^{-1}\left[g_1\left(t;\mathcal{F}^{\flat}_{\mathbb{C}^3/\mathbb{Z}_3}\right)\right] = 9t - 9t^2 + \cdots . \tag{3.22}$$

This agrees exactly with the content of (3.16) which says that $\mathcal{F}^{\flat}_{dP_0}$ should be the incomplete (since the plethystic logarithm does not terminate) intersection of 9 quadrics in 9 variables.

Now, we would like to refine the Hilbert series to include all the 5 global charges. From (3.19), we recognize the 9 quiver fields as transforming in the $[1,0]\times[0,1]$ representation of $SU(3)\times SU(3)$. For short, we denote an irreducible representation of this group as a collection of 4 nonnegative integer numbers, here $[1,0;0,1]$ and with obvious extension to other representations. We end up with the refinement of the Hilbert series for $\mathcal{F}^{\flat}_{\mathbb{C}^3/\mathbb{Z}_3}$ as

$$\begin{aligned} &g_1\left(t,x_1,x_2,y_1,y_2;\mathcal{F}^{\flat}_{\mathbb{C}^3/\mathbb{Z}_3}\right) \\ &\quad = \left(1-[0,1;1,0]t^2+([1,1;0,0]+[0,0;1,1])t^3-[1,0;0,1]t^4+t^6\right)\mathrm{PE}[[1,0;0,1]t]. \end{aligned} \tag{3.23}$$

The Hilbert series admits a simple and natural series expansion of the form

$$g_1\left(t,x_1,x_2,y_1,y_2;\mathcal{F}^{\flat}_{\mathbb{C}^3/\mathbb{Z}_3}\right) = \sum_{n=0}^{\infty}[n,0;0,n]t^n. \tag{3.24}$$

The Mesonic Moduli Space

The master space is a $(\mathbb{C}^*)^2$ fibration over the mesonic moduli space. The Hilbert series for the latter can be computed by integrating the Hilbert series of the master space in the directions corresponding to the $U(1)$ actions, which are weighted by y_1 and y_2:

$$g_1\left(t,x_1,x_2;{}^{\mathrm{mes}}\mathcal{M}=\frac{\mathbb{C}^3}{\mathbb{Z}_3}\right) = \oint_{|y_1|=1}\frac{dy_1}{2\pi i y_1}\oint_{|y_2|=1}\frac{dy_2}{2\pi i y_2}g_1\left(t,x_1,x_2,y_1,y_2;\mathcal{F}^{\flat}_{\mathbb{C}^3/\mathbb{Z}_3}\right) = \sum_{n=0}^{\infty}[3n,0]t^{3n}. \tag{3.25}$$

In the case of just one charge, setting the nonAbelian fugacities to 1, the Hilbert series simplifies to

$$g_1\left(t;{}^{\mathrm{mes}}\mathcal{M}=\frac{\mathbb{C}^3}{\mathbb{Z}_3}\right) = \frac{1+7t^3+t^6}{(1-t^3)^3}. \tag{3.26}$$

We see that the mesonic moduli space is a toric variety of dimension three, with a nonAbelian $SU(3)$ symmetry.

We can easily see that the Hilbert series (3.25) is the expected Hilbert series for $\mathbb{C}^3/\mathbb{Z}_3$. The orbifold $\mathbb{C}^3/\mathbb{Z}_3$ is obtained by modding $\mathbb{C}^3$ with coordinates z_1, z_2, z_3 by the discrete group $\mathbb{Z}_3$

$$z_i \longrightarrow \omega z_i, \quad i = 1, 2, 3, \qquad \omega^3 = 1. \tag{3.27}$$

By writing the ten basic invariants $Q_{ijk} = z_i z_j z_k$, we see that we can write $\mathbb{C}^3/\mathbb{Z}_3$ as the noncomplete intersection of 27 quadrics in $\mathbb{C}^{10}$. The generators Q_{ijk} are completely symmetric in the indices and transform in the $[3,0]$ representation of $SU(3)$. The holomorphic functions on $\mathbb{C}^3/\mathbb{Z}_3$ are given by all polynomials that are invariant under the $\mathbb{Z}_3$ action; these can be constructed as symmetrized products of z_i and there is one such holomorphic function for each $[3n, 0]$ irreducible representation of $SU(3)$. We then see that, if we assign fugacity t to the z_i, the Hilbert series of $\mathbb{C}^3/\mathbb{Z}_3$ correctly reproduces (3.25). In field theory, the generators Q_{ijk} correspond to the mesonic gauge-invariant operators $U_i V_j W_k$; the latter are completely symmetric in the indices precisely due to the F-term relations (3.16).

The Molien Invariant

Since the mesonic moduli space is an orbifold by a discrete symmetry group $\mathbb{Z}_3$, it is possible to compute its Hilbert series by using the Molien invariant. See details on this computation in [7, 9]. The action of the orbifold group on the 3 fugacities is $t_i \to \omega t_i, \omega^3 = 1$, and we take the average of the orbit over the orbifold group,

$$\begin{aligned} &g_1\left(t_1, t_2, t_3; \frac{\mathbb{C}^3}{\mathbb{Z}_3}\right) \\ &= \frac{1}{3}\left(\frac{1}{(1-t_1)(1-t_2)(1-t_3)} + \frac{1}{(1-\omega t_1)(1-\omega t_2)(1-\omega t_3)} + \frac{1}{(1-\omega^2 t_1)(1-\omega^2 t_2)(1-\omega^2 t_3)}\right) \\ &= \frac{1 + t_1^2 t_2 + t_2^2 t_3 + t_3^2 t_1 + t_1^2 t_3 + t_2^2 t_1 + t_3^2 t_2 + t_1 t_2 t_3 + t_1^2 t_2^2 t_3^2}{(1-t_1^3)(1-t_2^3)(1-t_3^3)}, \end{aligned} \tag{3.28}$$

which agrees with the expression in (3.25).

3.1.4. The General Case

We now discuss some general features of the master space for D3-branes at a conical CalabiYau singularity [14]. In the case of a toric CalabiYau $\mathcal{X}$ there is quite a general description of the associated quiver using Tilings [18, 19] and we mostly focus on this case. Many specific examples are discussed in details in [14].

In the case of a toric CalabiYau, we have [14] the following.

(i) $\mathcal{F}^{\flat}$ is a toric variety of complex dimension $g+2$, where g is the number of gauge groups. This is so because $\mathcal{X} \simeq \mathcal{F}^{\flat}//U(1)^{g}$, and an overall $U(1)$ decouples; thus $3 = \dim(\mathcal{F}^{\flat}) - (g-1)$. It is toric since it is acted upon by exactly $g+2\mathbb{C}^{*}$-actions corresponding to the classical global symmetries of the gauge theory: one R and two flavor, coming from the isometries of the toric threefold $\mathcal{X}$, as well as $g-1$ baryonic, IR relic symmetries of the nontrivial $U(1)$ factors, some of which are anomalous.

(ii) In the previously considered examples the moduli space was irreducible, but this is not the general case. The moduli space of gauge theories is well-known to have many branches and $\mathcal{F}^{\flat}$ is typically a reducible algebraic variety. This generalizes the familiar case of $\mathcal{N}=2$ gauge theories, where the moduli space is divided into a Coulomb branch and a Higgs branch. In the toric case, $\mathcal{F}^{\flat}$ contains a top-dimensional irreducible component of the same dimension, dubbed the *coherent component* and denoted as ${}^{\mathrm{Irr}}\mathcal{F}^{\flat}$, as well as many smaller-dimensional irreducible (generically) linear pieces, realized as coordinate hyperplanes.

(iii) The coherent component ${}^{\mathrm{Irr}}\mathcal{F}^{\flat}$ is itself an affine CalabiYau (We should better say that it is Gorenstein.) of dimension $g+2$. This is related to an intriguing property of the Hilbert Series for ${}^{\mathrm{Irr}}\mathcal{F}^{\flat}$. The numerator $P(t)$ of $g(t;{}^{\mathrm{Irr}}\mathcal{F}^{\flat})$, which we recall to be an integer polynomial of degree, say, n, has a palindromic symmetry for its coefficients $a_{j=0,\dots,n}$: $P(t)$ is invariant under the exchange $a_j \leftrightarrow a_{n-j}$. This is certainly true for the Hilbert series (3.10) and (3.21) and it is, in general, a consequence of a theorem by Stanley [33] and the fact that ${}^{\mathrm{Irr}}\mathcal{F}^{\flat}$ is CalabiYau. Interestingly, the same palindromic property of the Hilbert series is shared by the classical moduli space of SQCD for all values of N_c and N_f [4]; indeed the moduli space of SQCD is CalabiYau.

(iv) A detailed description of ${}^{\mathrm{Irr}}\mathcal{F}^{\flat}$ as algebraic variety is provided in [14]. $\mathcal{F}^{\flat}$ and ${}^{\mathrm{Irr}}\mathcal{F}^{\flat}$ can be written as a set of algebraic equations in $\mathbb{C}^{E}$, where E is the number of elementary fields. Alternatively, one can provide a symplectic quotient description of the toric variety which is useful for computing the Hilbert series. The latter description is directly related to the Dimer model of the quiver and the intriguing fact that perfect matchings generate the coherent component. We refer to [14] for details on the algebraic properties of $\mathcal{F}^{\flat}$ and ${}^{\mathrm{Irr}}\mathcal{F}^{\flat}$, the computation of their Hilbert series and the relation to Dimers.

(v) It is known that a given toric CalabiYau corresponds to many different quivers which are related by Seiberg duality. The set of such quivers where all gauge groups have the same number of colors is known as the set of toric phases of the given theory. For $N > 1$, all these different nonAbelian gauge theories are equivalent under Seiberg duality and flow to the same IR fixed point. For $N = 1$ there is no precise sense in which Seiberg duality can be defined, since there is no nonAbelian gauge group. Nevertheless we expect to see some relic of this duality. And in fact, although not isomorphic, the coherent component of the master space for different toric phases are closely related: their generating functions coincide, not only as a function of the R-symmetry parameter t but also when refined with all the non anomalous charge parameters. This follows from the fact that, as we discuss in the following, the coherent component of the master space determines the BPS spectrum for $N > 1$. Dual theories have the same non anomalous symmetries and the same spectrum of BPS operators.

Many other properties of the master space including the structure of the linear components, the relation to RG flows, and the existence of hidden symmetries are discussed in [14]. In the next section we see that perhaps the most important property of the master space is the fact that it determines the generating function for chiral operators for any value of N.

3.2. The Case of Many Branes

As we already said, the world-volume theory for N D3-branes is a quiver theory with product $U(N)$ gauge groups and, in the IR, the $U(1)$ factors decouple since only the special unitary groups exhibit asymptotic freedom and are strongly coupled in the IR. Thus the moduli space of interest is the space of solutions to the F-flatness, modded out by a nonAbelian gauge group

$$\mathcal{M}_N = \frac{\mathcal{F}_N^{\flat}}{\left(SU(N_1) \times \cdots \times SU(N_g)\right)}, \tag{3.29}$$

where the index N recalls that we are dealing with N branes. The moduli space $\mathcal{M}_N$ is of difficult characterization since the quotient is fully nonAbelian and it cannot be described by toric methods, as in the $N = 1$ case.

The mesonic moduli space is obtained from the full moduli space by performing a further quotient by the Abelian symmetries:

$$^{\text{mes}}\mathcal{M}_N \equiv \text{Sym}^N \mathcal{X} \simeq \mathcal{M}_N // U(1)^{g-1}. \tag{3.30}$$

For D3-branes at singularities, the mesonic moduli space should reproduce the motion of N mutually BPS D3-branes on the CalabiYau and it should be given by the symmetric product of N copies of $\mathcal{X}$. We see that the mesonic moduli space, for $\mathcal{X}$ being a CalabiYau threefold, is of dimension $3N$. From this result we can infer the dimension of the full moduli space $\mathcal{M}_N$ which must be $3N + g - 1$ for general N.

In the next section we discuss the main features of the nonAbelian moduli space using the examples of $\mathbb{C}^3$ and the conifold $\mathcal{C}$. We need some Plethysm [7]. To obtain the partition function $g_N(t)$ for $\text{Sym}^N \mathcal{X}$ from the partition function $g_1(t)$ for $\mathcal{X}$, one expands in power series the ν-inserted plethystic exponential

$$\text{PE}\left[\nu g_1(t_1, \ldots, t_n)\right] = \exp\left(\sum_{r=1}^{\infty} \frac{\nu^r g_1(t_1^r, \ldots, t_n^r)}{r}\right) = \sum_{N=0}^{\infty} g_N(t_1, \ldots, t_n)\nu^N. \tag{3.31}$$

3.2.1. The Case of $\mathbb{C}^3$

As we have seen, the master space $\mathcal{F}^{\flat}$ in the case of one brane coincides with the CalabiYau manifold $\mathcal{X} = \mathbb{C}^3$. In the nonAbelian case, the fields Φ_i are N by N matrices, transforming in the adjoint representation of the $U(N)$ gauge group. The F-terms require that the adjoint fields Φ_i commute:

$$\left[\Phi_i, \Phi_j\right] = 0. \tag{3.32}$$

The commuting matrices Φ_i can be simultaneously diagonalized using the action of the gauge group. The nonAbelian problem is thus reduced to N copies of the Abelian one, parameterized by the eigenvalues of the matrices Φ_i. As usual, the Weyl group of $U(N)$ survives as a discrete gauge symmetry after diagonalization. It acts as the group of permutation of the N Abelian copies of $\mathbb{C}^3$. As a result, the nonAbelian moduli space is the symmetric product $\mathrm{Sym}^N\mathbb{C}^3$. The Abelian part of the group still acts trivially on the adjoint fields, and therefore the moduli space is purely mesonic.

The generating function for $\mathrm{Sym}^N\mathbb{C}^3$ can be extracted from the term of order ν^N in the ν-inserted Plethystic exponential

$$g\left(\nu;t_1,t_2,t_3;\mathbb{C}^3\right) = \mathrm{PE}[\nu g_1] = \sum_{N=0}^{\infty} g_N\left(t_1,t_2,t_3;\mathbb{C}^3\right)\nu^N, \quad g_1\left(t_1,t_2,t_3;\mathbb{C}^3\right) = \frac{1}{(1-t_1)(1-t_2)(1-t_3)}. \tag{3.33}$$

The full expression can be explicitly written as 3 infinite products

$$g\left(\nu;t_1,t_2,t_3;\mathbb{C}^3\right) = \prod_{n_1=0}^{\infty}\prod_{n_2=0}^{\infty}\prod_{n_3=0}^{\infty}\frac{1}{1-\nu t_1^{n_1}t_2^{n_2}t_3^{n_3}}, \tag{3.34}$$

which coincides with the grand-canonical partition function of the three dimensional harmonic oscillator.

All the generating functions $g_N(t_1,t_2,t_3;\mathbb{C}^3)$ can be expanded in terms of characters of the $U(3)$ global symmetry. Some examples are reported in [14].

3.2.2. The Case of the Conifold

The case of the conifold is more interesting since it has a baryonic branch.

The Mesonic Moduli Space

The IR CFT is described by a quiver with two gauge groups $SU(N)\times SU(N)$ and four chiral bifundamental fields, $A_{1,2}$ transforming in the $(N,\overline{N})$ representation and $B_{1,2}$ transforming in the $(\overline{N},N)$ representation, interacting with the quartic superpotential $\epsilon_{ij}\epsilon_{pq}A_iB_pA_jB_q$. The charges of the fields under the global symmetries and the corresponding fugacities are summarized in Table 1. The mesonic gauge-invariant operators are those neutral under $U(1)_B$. We have four basic ones:

$$z_{ij} = \mathrm{Tr}\left(A_iB_j\right). \tag{3.35}$$

By relaxing the trace in this expression, we can consider the $z_{ij} = A_iB_j$ as $N\times N$ matrices transforming in the adjoint of the first gauge group. It is easy to check that the F-terms imply that z_{ij} commute and satisfy the matrix equation

$$z_{11}z_{22} - z_{12}z_{21} = 0 \tag{3.36}$$

which corresponds to the defining equation for the conifold singularity. The mutually commuting z_{ij} can be simultaneously diagonalized by an adjoint action of the first gauge group. Due to the matrix equation (3.36), the eigenvalues must satisfy the conifold equation. As in the case of $\mathbb{C}^3$, we reduce the problem to N copies of the Abelian one. Finally, using the remaining Weyl permutation symmetry, we see that the mesonic moduli space is indeed $Sym^N \mathcal{C}$.

Now it is very simple to write the generating function $g_N(t,x,y;{}^{mes}\mathcal{M}_N)$ for the mesonic moduli space using the Plethystic Exponential [7]. Since ${}^{mes}\mathcal{M}_N \equiv Sym^N \mathcal{X}$, we conclude from the previous discussion that g_N is obtained from g_1, computed in (3.12), by applying the Plethystic Exponential:

$$\begin{aligned} \mathrm{PE}\left[\nu g_1(t,x,y;\mathcal{C})\right] &= \sum_{N=0}^{\infty} g_N\left(t,x,y;{}^{\mathrm{mes}}\mathcal{M}_N\right)\nu^N, \\ g_1 &= \frac{1-t^4}{(1-t^2xy)(1-t^2x/y)(1-t^2y/x)(1-t^2/xy)}. \end{aligned} \tag{3.37}$$

The Baryonic Branch

The plethystic program can be efficiently applied also to the study of the baryonic branch [12]. As we saw, the generating function for the master space is freely generated by the four basic fields of the conifold gauge theory and it takes the form

$$g_1\left(t_1,t_2,x,y;\mathcal{F}^{\flat}_{\mathcal{C}}\right) = \frac{1}{(1-t_1x)(1-t_1/x)(1-t_2y)\left(1-\dfrac{t_2}{y}\right)}. \tag{3.38}$$

In the following, we set $x = y = 1$ for simplicity. General formulae including the $SU(2)$ chemical potentials can be found in [11]. We can decompose g_1 into sectors with fixed baryonic charge B, each with multiplicity one:

$$\begin{aligned} g_1\left(t_1,t_2;\mathcal{F}^{\flat}_{\mathcal{C}}\right) &= \sum_{B=-\infty}^{\infty} g_{1,B}\left(t_1,t_2;\mathcal{F}^{\flat}_{\mathcal{C}}\right), \\ g_{1,B>0}\left(t_1,t_2;\mathcal{F}^{\flat}_{\mathcal{C}}\right) &= \sum_{n=0}^{\infty}(n+1+B)(n+1)t_1^{n+B}t_2^{n}, \\ g_{1,B<0}\left(t_1,t_2;\mathcal{F}^{\flat}_{\mathcal{C}}\right) &= \sum_{n=0}^{\infty}(n+1)(n+1+|B|)t_1^{n}t_2^{n+|B|}. \end{aligned} \tag{3.39}$$

It is manifest that each term in $g_{1,B}$ has a monomial b^B corresponding to a baryonic charge B. The decomposition into each baryonic charge can be computed by expanding $g_1(t,b;\mathcal{C})$ in a formal Laurent series in b. $g_{1,B>0}$ contains all monomial $A_{i_1}B_{j_1}\cdots A_{i_{n+B}}B_{j_n}$ transforming in the $[n+B;n]$ representation of $SU(1)_1 \times SU(2)_2$, and analogously $g_{1,B<0}$ contains all $A_{i_1}B_{j_1}\cdots A_{i_n}B_{j_{n+|B|}}$ transforming in the $[n;n+|B|]$ representation. It is then obvious that the sum of all $g_{1,B}$ reconstructs the generating function of the free ring in the four generators A_1, A_2, B_1, B_2.

It is quite remarkable that the generating function for the entire moduli space is obtained by applying the Plethystic Exponential to each sector of definite baryonic charge. The result for generic N is indeed obtained as follows:

$$g\left(\nu; t_1, t_2; \mathcal{F}_{\mathcal{C}}^{\flat}\right) = \sum_{B=-\infty}^{\infty} \mathrm{PE}\left[\nu g_{1,B}\left(t_1, t_2; \mathcal{F}_{\mathcal{C}}^{\flat}\right)\right],$$
$$g\left(\nu; t_1, t_2; \mathcal{F}_{\mathcal{C}}^{\flat}\right) = \sum_{N=0}^{\infty} \nu^N g_N\left(t_1, t_2; \mathcal{F}_{\mathcal{C}}^{\flat}\right). \tag{3.40}$$

We can compute, for example, the generating function for $N = 2$, by taking the coefficient of ν^2 in this expression:

$$g_2(t_1, t_2; \mathcal{C}) = \frac{1 + t_1 t_2 + t_1^2 t_2^2 - 3t_1^4 t_2^2 - 3t_1^2 t_2^4 + t_1^5 t_2^3 + t_1^3 t_2^5 - 3t_1^3 t_2^3 + 4t_1^4 t_2^4}{\left(1 - t_1^2\right)^3 (1 - t_1 t_2)^3 \left(1 - t_2^2\right)^3}. \tag{3.41}$$

This expression can be written in a more revealing form by noting that the $U(4)$ symmetry of the master space, $\mathbb{C}^4$, remains the global symmetry for the special case where the number of color is $N = 2$ [14]. We therefore expect to expand the Hilbert series in terms of characters of irreducible representations of $U(4)$, where the $U(1)$ symmetry is the fugacity is denoted by t. This is computed in [14] and takes the form

$$g_2(t_1, t_2; \mathcal{C}) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} [2n_1, 2n_2, 0] t^{2n_1 + 4n_2}, \tag{3.42}$$

where the character of an irreducible representation of $SU(4)$ is denoted by $[n_1, n_2, n_3]$. An explicit analysis reveals that the moduli space is generated by the following operators, transforming in the $[2, 0, 0]$ representation of $SU(4)$ of dimension 10: 3 baryons det $A_i A_j$, 3 baryons det $B_i B_j$ and 4 mesons Tr $A_i B_j$. The structure of the generating functions for $N \geq 2$ is discussed in details in [11, 12].

Formula (3.41) and its analogs for small values of N can be compared with the result in the classical theory computed using computer algebra programs, like Macaulay2, and it turns out that there is perfect agreement [11, 12].

Formula (3.40) is motivated by a computation in the gravitational dual, where the BPS states can be represented as states of supersymmetric D3-branes [8, 34]. A review of the dual computation is too detailed and we refer to [8] for details. Let us notice few things, however.

(i) BPS states in the gravitational AdS dual can be obtained by quantizing classical configurations of supersymmetric D3-branes. The latter fall in classes according to the homology of the cycle they are wrapping. It can be shown that the baryonic quantum number B is in correspondence with the homology number of the base $T^{1,1}$ of the conifold. The (geometric) quantization of classical supersymmetric D3-branes is done in each sector independently and it amounts to taking the Plethystic Exponential in each sector [8, 34, 35]. This explains the conjecture in formula (3.40).

(ii) By quantizing D3-branes, we can perform a computation valid at strong coupling and large N. In fact, $g_{1,B}$, which is is the generating function for sections of a given line bundle on $\mathcal{C}$, can be computed in a purely geometric way using the equivariant Atiyah-Singer theorem [8, 36]. Quite remarkably, the computation agrees with (3.39) and the large N strong coupling prediction for the BPS generating function agrees with a computation performed in the classical theory for small N.

(iii) Formula (3.40) is intriguing and once more shows the deep connection of the quiver gauge theory with the geometry: the left hand side, which is computed in the quiver theory using F-terms, has an expansion as a sum of functions that can be computed using the geometry of the CalabiYau only.

3.2.3. *The General Case*

Here we try to summarize what we have learned from the previous examples.

In general, for all quivers describing D3-branes at conical singularities, we must have ${}^{\mathrm{mes}}\mathcal{M}_N \equiv \mathrm{Sym}^N \mathcal{X}$, and this can be explicitly proven in the case of quivers based on Tilings. As a consequence of this identification, the generating function $g_N(t_i, {}^{\mathrm{mes}}\mathcal{M}_N)$ for the mesonic moduli space can be always written using the Plethystic Exponential [7],

$$\mathrm{PE}\left[\nu g_1(t_i, {}^{\mathrm{mes}}\mathcal{M}_N)\right] = \sum_{N=0}^{\infty} g_N(t_i, {}^{\mathrm{mes}}\mathcal{M}_N)\nu^N. \tag{3.43}$$

More delicate is the case of the generating function of the entire moduli space $\mathcal{M}$, including the baryonic directions. In the case of the conifold the previous discussion gives a remarkably simple and neat answer. The situation is, in general, more complicated. In all the examples examined in [12], it seems true that there exists a decomposition of the generating function for the master space into sectors with definite baryonic charge

$$g_1\left(t_i; {}^{\mathrm{Irr}}\mathcal{F}^{\flat}\right) = \sum_{B} g_{1,B}(t_i), \tag{3.44}$$

such that the generating function for the entire moduli space is obtained by applying the Plethystic Exponential to each sector,

$$\sum_{B} \mathrm{PE}\left[\nu g_{1,B}(t_i)\right] = \sum_{N=0}^{\infty} \nu^N g_N(t_i). \tag{3.45}$$

In the previous formula, the sum over B in (3.44) is in general a sum over a lattice of charges, related to the GKZ decomposition of the moduli space of resolution of the CalabiYau. We refer to [12] for examples and discussion.

4. Membranes at Singularities

Quiver theories play also an important role in understanding the world-volume theories for M2 branes probing four-dimensional CalabiYau singularities $\mathcal{Y}$. These are the holographic duals of backgrounds of the form $AdS_4 \times H_7$, where H_7 is the Sasaki-Einstein seven dimensional base of $\mathcal{Y}$. The construction of the dual field theory has been an unsolved problem for long time. It has been recently understood that $\mathcal{N} = 2$ Chern-Simons gauge theories [24–26] play an important role in this question and give rise to theories with the right properties. The construction generalizes the proposal for the maximally supersymmetric case [20].

We then consider quiver theories $\mathcal{N} = 2$ theories in 2+1 dimensions with $U(N)$ gauge groups and adjoint and bifundamental chiral matter superfields Φ_{ab} interacting through a superpotential $W(\Phi_{ab})$. We use the letter $a = 1, \ldots, g$ to label the gauge groups. With standard Yang-Mills interactions, the moduli space is obtained by solving the F and D term constraints (We use the same symbol to denote the scalar $\mathcal{N} = 2$ superfields and their lowest components.)

$$\begin{gathered} \partial_{\Phi_{ab}} W = 0, \\ \mathfrak{D}_a(\Phi) \equiv \sum_b \Phi_{ab}\Phi_{ab}^{\dagger} - \sum_c \Phi_{ca}^{\dagger}\Phi_{ca} + \left[\Phi_{aa}, \Phi_{aa}^{\dagger}\right] = 0 \end{gathered} \tag{4.1}$$

and dividing by the gauge group. We can then determine the moduli space as discussed in the previous Section. The master space $\mathcal{F}^{\flat}$ is a $g - 1$ fibration over the mesonic moduli space $\mathcal{X}$. Let us only consider quivers where $\mathcal{X}$ is a CalabiYau threefold.

Consider now the same theory without Yang-Mills kinetic term for the gauge groups but with a Chern-Simons interaction with integer coefficients k_a, satisfying $\sum k_a = 0$. The moduli space is now bigger [24–26]. In $\mathcal{N} = 2$ supersymmetry in three dimensions the gauge vector has a scalar partner σ, which, in a Chern-Simons theory, has no kinetic term and it is an auxiliary field. The bosonic potential is

$$\sum_{\Phi_{ab}} \mathrm{Tr}\Big((\sigma_a \Phi_{ab} - \Phi_{ab}\sigma_b)(\sigma_a \Phi_{ab} - \Phi_{ab}\sigma_b)^{\dagger} + |\partial_{\Phi_{ab}} W|^2 \Big), \tag{4.2}$$

where the auxiliary fields are determined by the constraints

$$\mathfrak{D}_a(\Phi) = \frac{k_a}{2\pi}\sigma_a. \tag{4.3}$$

The potential is minimized by

$$\begin{gathered} \partial_{\Phi_{ab}} W = 0, \\ \sigma_a \Phi_{ab} - \Phi_{ab}\sigma_b = 0. \end{gathered} \tag{4.4}$$

In the Abelian case, all $\sigma_a \equiv \sigma$ are equal and the equations $\mathfrak{D}_a(\Phi) = (k_a/2\pi)\sigma$ reduce to the standard D-terms of an $\mathcal{N} = 2$ theory with a FI term depending on the Chern-Simons couplings. Since $\sum_a k_a = 0$ and $\sum_a \mathfrak{D}_a(\Phi) = 0$ by construction, one of these equations is redundant. Moreover, any linear combination of gauge groups with coefficient m_a orthogonal to the CS parameters $\sum_a k_a m_a = 0$ has a vanishing moment map. We are thus imposing $g - 2$ D-term constraints, where g is the number of gauge groups. We can impose simultaneously the D-term constraints and the corresponding $U(1)$ gauge transformations by modding out by the complexified gauge group. We do not need to impose the last D term condition since it determines the value of the auxiliary field σ. Moreover, the corresponding $U(1)$ group, through its CS coupling with the overall gauge field, it is broken to $\mathbb{Z}_k$, where $k = \gcd(\{k_a\})$ [20, 37, 38]. As a result the Abelian moduli space has a dimension of one unit bigger than $\mathcal{X}$ and it has the general form $\mathcal{Y}/\mathbb{Z}_k$ [24–26]. We see that $\mathcal{Y}$ can be obtained from the master space by dividing by $g - 2$ complexified $U(1)$ groups. In the interesting case where $\mathcal{X}$ was a threefold, we obtain a four-dimensional manifold. For toric quiver based on Tilings, and some other generalization, one can explicitly show that the moduli space is still CalabiYau [26].

Modding by the remaining complexified $U(1)$, which in the membrane theory is broken to $\mathbb{Z}_k$, we would obtain the threefold $\mathcal{X}$. We thus see that $\mathcal{Y}$ is in general a $\mathbb{C}^*$ fibration over $\mathcal{X}$. $\mathcal{X}$ is uniquely specified by the gauge group and matter field of the quiver, while $\mathcal{Y}$ is specified also by the choice of Chern-Simons couplings k_a. By varying k_a we obtain a family of $\mathbb{C}^*$ fibrations over Z. We thus have the chain of fibrations

$$\mathcal{F}^b \xrightarrow[(\mathbb{C}^*)^{g-2}]{} \mathcal{Y} \xrightarrow[\mathbb{C}^*]{\{k_a\}} \mathcal{X} \tag{4.5}$$

which is determined by a choice of a particular $\mathbb{C}^*$ direction in the master space depending on the Chern-Simons couplings k_a.

The non Abelian case requires some care, as not all operators are manifestly gauge-invariant. It is conjectured in the literature that this difficulty is solved by the introduction of so called "monopole operators" which are special objects in 2+1 dimensions. This proposal states that the chiral operators in the Lagrangian are dressed by monopole operators such that the collection of generators on the moduli space become a set of commuting adjoint valued operators and then this naturally leads to symmetric products of the CY4 singularity. To date there have not been satisfactory solutions to this problem. There is a simple argument which supports the conjecture that the moduli space for higher values of N is the symmetric product and it goes as follows. By a gauge transformation, we can diagonalize all the σ_a. The equations $\sigma_a \Phi_{ab} = \Phi_{ab} \sigma_b$ tell you that there is a branch where (generically) all the fields Φ_{ab} are diagonal. This reduces the problem to N copies of the Abelian one. The remaining discrete gauge symmetry corresponding to the Weyl group of $SU(N)$ implies that the moduli space is generically the N-fold symmetric product of the Abelian one. We see that the Chern-Simons theory nicely enforces in 2+1 dimensions a structure of the moduli space which is very natural from the point of view of M2 branes. It is possible, as in 3+1 dimensions, that the moduli space for some particular quiver contains various different branches of the moduli space. Henceforth we avoid these subtle issues and always refer to the branch corresponding to the symmetric product.

Let us examine some very simple examples. Many others can be found in [24–27, 39–50].

4.1. A Case Based on the Conifold Quiver (ABJM)

We need at least two gauge groups to enforce the condition $\sum_a k_a = 0$. A quick look at the classification of brane Tilings in [47] shows that there are two theories with 2 gauge groups, the 2-square model and the 2-hexagon model. We also need a quartic superpotential, as this is the natural scale invariant interaction term in 2+1 dimensions. This selects the 2-square model which corresponds to the conifold in 3+1 dimensions. Thus, we have the conifold quiver with two gauge groups and opposite Chern-Simons terms $U(N)_k \times U(N)_{-k}$, fields $A_i, B_i, i = 1, 2$ transforming in the $(N, \overline{N})$ and $(\overline{N}, N)$ representation of the gauge group, respectively, and interacting with the superpotential

$$W = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1. \tag{4.6}$$

We call this theory $\widetilde{\mathcal{C}}_{\{k,-k\}}$.

It is by now well known that this theory, dubbed ABJM theory, has $\mathcal{N} = 6$ supersymmetry and it is the candidate dual for the background $AdS_4 \times S^7/\mathbb{Z}_k$ [20]. This can be seen by studying the Abelian moduli space. The master space for the conifold theory is $\mathbb{C}^4$, as discussed in Section 3.1.2. Since the number of gauge groups is $g = 2$ there is no complexified gauge group to mod by. The non trivial $U(1)$ is broken to $\mathbb{Z}_k$ and we recover the result [20] that the moduli space for the theory is $\mathbb{C}^4/\mathbb{Z}_k$, with an orbifold action $(1, 1, -1, -1)$.

For $k = 1$, the Hilbert series takes a simple form,

$$g\left(t, \widetilde{\mathcal{C}}_{\{1,-1\}}\right) = \frac{1}{(1-t)^4}. \tag{4.7}$$

The generators are A_1, A_2, B_1, B_2 corresponding to the fundamental fields of the conifold theory.

The moduli space for higher k is then given by the $\mathbb{Z}_k$ action $1, 1, -1, -1$ on the generators, respectively. Let us compute the Hilbert Series for this model by averaging on the discrete group using the discrete Molien formula

$$\begin{aligned} g\left(t, \widetilde{\mathcal{C}}_{\{k,-k\}}\right) &= \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{(1-\omega^j t)^2 (1-\omega^{-j} t)^2} \\ &= \frac{1 - t^4 + 2kt^k - 4kt^{k+2} + 2kt^{k+4} - t^{2k} + t^{2k+4}}{(1-t^2)^4 (1-t^k)^2} \\ &= \frac{1 + t^2 + 2kt^k - 2kt^{k+2} - t^{2k} - t^{2k+2}}{(1-t^2)^3 (1-t^k)^2}, \end{aligned} \tag{4.8}$$

with $\omega^k = 1$. The generators are now $A_i B_j$ and A^k, B^k, as indicated by the denominator of the first expression. We also see that the Hilbert series is palindromic, as is the master space in 3+1 dimensions, and it has a pole of order 4 at $t = 1$, indicating that the moduli space is a CalabiYau fourfold.

For large values of k the Hilbert Series behaves like

$$g\left(t,\widetilde{\mathcal{C}}_{\{k,-k\}}\right)=\frac{1+t^2}{\left(1-t^2\right)^3}\left(1+O\left(t^k\right)\right), \tag{4.9}$$

which is the Hilbert series for the conifold. The large k limit is equivalent to dividing the four-dimensional moduli space by the $\mathbb{C}^*$ action specified by the charges under the remaining $U(1)$ gauge group. We see that, as discussed above, the four-dimensional CalabiYau is a $\mathbb{C}^*$ fibration over the conifold.

4.1.1. A Remark: The Extended Supersymmetry of the ABJM Model

The Hilbert series (4.8) counts the holomorphic functions on the fourfold CalabiYau $\mathbb{C}^4/\mathbb{Z}_k$. These can be also set in one-to-one correspondence with the $\mathcal{N}=2$ chiral KK supermultiplets arising in the compactification on $S^7/\mathbb{Z}_k$. However, the ABJM models have extended $\mathcal{N}=6$ supersymmetry and the $\mathcal{N}=2$ chiral supermultiplets are actually part of bigger $\mathcal{N}=6$ chiral multiplets. It is then of some interest to write a generating function for $\mathcal{N}=6$ chiral multiplets on $AdS_4\times S^7/\mathbb{Z}_k$ [52].

Let us first consider the case $k=1$ with $\mathcal{N}=8$ supersymmetry. The chiral multiplets are actually in one-to-one correspondence with harmonic functions on S^7, which can be written as symmetric traceless tensors and transform in the $[n,0,0,0]$ representation of the $SO(8)$ global symmetry. The generating function for all harmonic functions on S^7 is then

$$g_1\left(t;S^7\right)=\left(1-t^2\right)\mathrm{PE}\left[[1,0,0,0]_{SO(8)}t\right]=\sum_{n=0}^{\infty}[n,0,0,0]_{SO(8)}t^n. \tag{4.10}$$

The role of $(1-t^2)$ is to remove traces from the symmetric product expansion. An extension of this formula to include the spin degrees of freedom under the little group in 3+1 dimensions is as follows. Introduce the fugacity x for and the formula takes the form

$$\frac{(1-t^2)}{x^4}\mathrm{PE}\left[[1,0,0,0]_{SO(8)}t\right]\mathrm{PE}_F\left[[0,0,0,1]_{SO(8)}x\right], \tag{4.11}$$

but will not be discussed further. The result for higher values of k is obtained by using the discrete Molien invariant. The action of $\mathbb{Z}_k\subset U(1)$ breaks $SO(8)$ to $SU(4)\times U(1)$, where $SU(4)$ is the R-symmetry for $\mathcal{N}=6$ supersymmetry in 2+1 dimensions. For this purpose we introduce the fugacity b for $U(1)$, and decompose the 8 dimensional representation of $SO(8)$ into two irreducible representations of $SU(4)$:

$$[1,0,0,0]_{SO(8)}t=[1,0,0]_{SU(4)}t_1+[0,0,1]_{SU(4)}t_2, \tag{4.12}$$

where $t_1=tb, t_2=t/b$. Explicit expressions for the characters of the $SU(4)$ representations can be taken to be with 3 complex fugacities, z_1,z_2,z_3 in the form,

$$[1,0,0]=z_1+\frac{z_2}{z_1}+\frac{z_3}{z_2}+\frac{1}{z_3},\qquad [0,0,1]=\frac{1}{z_1}+\frac{z_1}{z_2}+\frac{z_2}{z_3}+z_3. \tag{4.13}$$

The partition function is then

$$g_1\left(t, b, z_1, z_2, z_3; \frac{S^7}{\mathbb{Z}_k}\right) = \frac{1}{k}\sum_{j=0}^{k-1} g_1\left(t, w^j b, z_1, z_2, z_3; S^7\right). \tag{4.14}$$

It is interesting to note that in the limit where k goes to infinity, all states with non zero baryonic charge disappear from the spectrum and we obtain the generating function

$$g_1\left(t, z_1, z_2, z_3; \mathbb{P}^3\right) = \sum_{n=0}^{\infty} [n, 0, n] t^{2n}, \tag{4.15}$$

where $[n, 0, n]$ denotes an $SU(4)$ representation.

In the limit where $k \to \infty$ the M theory compactification on $AdS_4 \times S^7/\mathbb{Z}_k$ can be effectively reduced to the Type-IIA compactification on $AdS_4 \times \mathbb{P}^3$. The generating function (4.15) is reinterpreted as the partition function for $\mathcal{N} = 6$ chiral multiplets in the KK compactification on $\mathbb{P}^3$, which indeed fall in $[n, 0, n]$ representations of $SU(4)$ [53].

The Hilbert series (4.8) and (4.9) can be analogously interpreted as the partition functions for the $\mathcal{N} = 2$ KK chiral multiplets on $AdS_4 \times S^7/\mathbb{Z}_k$ and $AdS_4 \times \mathbb{P}^3$, respectively. They differ from the equations (4.14) and (4.15) since we are counting only an $\mathcal{N} = 2$ subset of the protected operators in $\mathcal{N} = 6$ supersymmetry. For example, out of $\dim[n, 0, n] = (n+1)^2(n+2)^2(2n+3)/12$ protected operators in $\mathcal{N} = 6$ there are precisely $(n+1)^2$ operators which are holomorphic under the $\mathcal{N} = 2$ subgroup. We therefore sum

$$\sum_{n=0}^{\infty} (n+1)^2 t^{2n} = \frac{1+t^2}{(1-t^2)^3}, \tag{4.16}$$

and get the result computed in (4.9).

4.2. *A Case Based on the* $\mathbb{C}^3/\mathbb{Z}_3$ *Quiver*

Another simple example [26] is based on the quiver for $\mathbb{C}^3/\mathbb{Z}_3$ with Chern-Simons couplings $(k_1, k_2, -k_1, -k_2)$. As we saw in Section 3.1.3, the master space has dimension five. The moduli space is given by modding the five-dimensional master space by the $U(1)$ gauge symmetry prescribed by the CS terms. We find a two parameter family of CalabiYau fourfolds that can be identified with the cone over the manifolds $Y^{p,k}(\mathbb{C}P^2)$ [25, 54].

We focus, for simplicity, on the case of CS parameters $k_1 = k_2 = 1$. The corresponding fourfold is the cone over the coset (This is sometimes called $M^{3,2}$.) manifold $M^{1,1,1} = SU(3) \times SU(2) \times U(1)/SU(2) \times U(1) \times U(1)$ with global symmetry $SU(3) \times SU(2) \times U(1)_R$. The coincidence of this global symmetry with the gauge group of the standard model for particle interactions was a reason for enhanced activity back in the 80's. The Calabi Yau fourfold is then obtained by modding out by the gauge group $U(1)_1 - U(1)_2$ which, as seen from equation (3.17) corresponds to the action $\{0, 0, 0, 2, -1, -1\}$ on perfect matchings and it breaks the global symmetry on the master space from $SU(3) \times SU(3) \times U(1)$ to $SU(3) \times SU(2) \times U(1)$. Note that part of the hidden symmetry now becomes a symmetry of the mesonic moduli space in the 2+1 dimensional theory. By setting $y_1 = wx$, $y_2 = w^2$ in (3.20), x becomes the fugacity for

$SU(2)$ and w is the fugacity for the $U(1)$ which is gauged. The integration over w leaves 4 fugacities t, x_1, x_2, x which parametrize the four toric symmetries of the CalabiYau:

$$\begin{aligned} g&\left(t, x, y, \tilde{x}; \widetilde{\mathbb{C}^3/\mathbb{Z}_3}\right) \\ &= \oint \frac{dz}{2\pi iz}\frac{dw}{2\pi iw}\frac{1}{(1-tx_1/z)\ (1-tx_2/x_1z)(1-t/x_2z)(1-w^2z)(1-zx/w)(1-z/wx)} \\ &= \sum_{k=0}^{\infty}[3k,0;2k]t^{3k}, \end{aligned} \tag{4.17}$$

where $[n,m;s]$ denotes irreps of $SU(3)\times SU(2)$. From the last expression we recognize indeed the KK spectrum of M theory compactified on $M^{1,1,1}$ [55].

The case of higher k is obtained by modding out by an action $\mathbb{Z}_k \subset U(1)_1 + U(1)_2 - 2U(1)_3$ which further breaks the global symmetry to $SU(3)\times U(1)\times U(1)$. The Hilbert series for $k > 1$ can be computed with the methods described above and is given in [48].

Acknowledgments

It is a pleasure to acknowledge Agostino Butti, Davide Forcella, Yang-Hui He, Noppadol Mekareeya and David Vegh for collaboration on the topics contained in this paper. The second author is supported in part by INFN and MIUR under contract no. 2007-5ATT78-002.

References

[1] P. Pouliot, “Molien function for duality,” *Journal of High Energy Physics*, vol. 3, no. 1, article 021, 1999.

[2] C. Römelsberger, “Counting chiral primaries in $N = 1$, $d = 4$ superconformal field theories,” *Nuclear Physics B*, vol. 747, no. 3, pp. 329–353, 2006.

[3] A. Hanany and N. Mekareeya, “Counting gauge invariant operators in SQCD with classical gauge groups,” *Journal of High Energy Physics*, vol. 2008, no. 10, article 012, 2008.

[4] J. Gray, Y.-H. He, A. Hanany, N. Mekareeya, and V. Jejjala, “SQCD: a geometric aperçu,” *Journal of High Energy Physics*, vol. 2008, no. 5, aricle 099, 2008.

[5] J. Kinney, J. Maldacena, S. Minwalla, and S. Raju, “An index for 4 dimensional super conformal theories,” *Communications in Mathematical Physics*, vol. 275, no. 1, pp. 209–254, 2007.

[6] D. Martelli and J. Sparks, “Dual giant gravitons in Sasaki-Einstein backgrounds,” *Nuclear Physics B*, vol. 759, no. 1-2, pp. 292–319, 2006.

[7] S. Benvenuti, B. Feng, A. Hanany, and Y.-H. He, “Counting BPS operators in gauge theories: quivers, syzygies and plethystics,” *Journal of High Energy Physics*, vol. 2007, no. 11, article 050, 2007.

[8] A. Butti, D. Forcella, and A. Zaffaroni, “Counting BPS baryonic operators in CFTs with Sasaki-Einstein duals,” *Journal of High Energy Physics*, vol. 2007, no. 6, article 069, 2007.

[9] B. Feng, A. Hanany, and Y. He, “Counting gauge invariants: the plethystic program,” *Journal of High Energy Physics*, vol. 2007, no. 3, article 090, 2007.

[10] A. Hanany and C. Römelsberger, “Counting BPS operators in the chiral ring of $N = 2$ supersymmetric gauge theories or $N = 2$ braine surgery,” *Advances in Theoretical and Mathematical Physics*, vol. 11, no. 6, pp. 1091–1112, 2007.

[11] D. Forcella, A. Hanany, and A. Zaffaroni, “Baryonic generating functions,” *Journal of High Energy Physics*, vol. 2007, no. 12, article 022, 2007.

[12] A. Butti, D. Forcella, A. Hanany, D. Vegh, and A. Zaffaroni, “Counting chiral operators in quiver gauge theories,” *Journal of High Energy Physics*, vol. 2007, no. 11, article 092, 2007.

[13] D. Forcella, "BPS partition functions for Quiver Gauge theories: counting fermionic operators," http://arxiv.org/abs/0705.2989.

[14] D. Forcella, A. Hanany, Y.-H. He, and A. Zaffaroni, "The master space of $N = 1$ gauge theories," *Journal of High Energy Physics*, vol. 2008, no. 8, article 012, 2008.

[15] I. R. Klebanov and E. Witten, "Superconformal field theory on threebranes at a Calabi-Yau singularity," *Nuclear Physics B*, vol. 536, no. 1-2, pp. 199–218, 1998.

[16] B. S. Acharya, J. M. Figueroa-O'Farrill, C. M. Hull, and B. J. Spence, "Branes at conical singularities and holography," *Advances in Theoretical and Mathematical Physics*, vol. 2, pp. 1249–1286, 1999.

[17] D. R. Morrison and M. R. Plesser, "Non-spherical horizons. I," *Advances in Theoretical and Mathematical Physics*, vol. 3, pp. 1–81, 1999.

[18] A. Hanany and K. D. Kennaway, "Dimer models and toric diagrams," http://arxiv.org/abs/hep--th/0503149.

[19] S. Franco, A. Hanany, D. Vegh, B. Wecht, and K. D. Kennaway, "Brane dimers and quiver gauge theories," *Journal of High Energy Physics*, vol. 2006, no. 1, article 096, pp. 2401–2448, 2006.

[20] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, "$N = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," *Journal of High Energy Physics*, vol. 2008, no. 10, 2008.

[21] M. Benna, I. Klebanov, T. Klose, and M. Smedbäck, "Superconformal Chern-Simons theories and AdS_4/CFT_3 correspondence," *Journal of High Energy Physics*, vol. 2008, no. 9, article 072, 2008.

[22] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee, and J. Park, "$N = 5, 6$ superconformal Chern-Simons theories and M2-branes on orbifolds," *Journal of High Energy Physics*, vol. 2008, no. 9, article 002, 2008.

[23] O. Aharony, O. Bergman, and D. L. Jafferis, "Fractional M2-branes," *Journal of High Energy Physics*, vol. 2008, no. 11, article 043, 2008.

[24] D. L. Jafferis and A. Tomasiello, "A simple class of $N = 3$ gauge/gravity duals," *Journal of High Energy Physics*, vol. 2008, no. 10, article 101, 2008.

[25] D. Martelli and J. Sparks, "Moduli spaces of Chern-Simons quiver gauge theories and AdS_4/CFT_3," *Physical Review D*, vol. 78, no. 12, Article ID 126005, 2008.

[26] A. Hanany and A. Zaffaroni, "Tilings, Chern-Simons theories and M2 branes," *Journal of High Energy Physics*, vol. 2008, no. 10, article 111, 2008.

[27] A. Hanany, D. Vegh, and A. Zaffaroni, "Brane tilings and M2 branes," *Journal of High Energy Physics*, vol. 2009, no. 3, 2009.

[28] D. Forcella, A. Hanany, Y.-H. He, and A. Zaffaroni, "Mastering the master space," *Letters in Mathematical Physics*, vol. 85, no. 2-3, pp. 163–171, 2008.

[29] D. Amati, K. Konishi, Y. Meurice, G. C. Rossi, and G. Veneziano, "Non-perturbative aspects in supersymmetric gauge theories," *Physics Reports*, vol. 162, no. 4, pp. 169–248, 1988.

[30] F. Cachazo, M. R. Douglas, N. Seiberg, and E. Witten, "Chiral rings and anomalies in supersymmetric gauge theory," *Journal of High Energy Physics*, vol. 6, no. 12, article 071, pp. 1611–1666, 2002.

[31] D. Grayson and M. Stillman, "Macaulay 2, a software system for research in algebraic geometry," http://www.math.uiuc.edu/Macaulay2.

[32] A. Hanany, N. Mekareeya, and G. Torri, "The Hilbert series of adjoint SQCD," *Nuclear Physics B*, vol. 825, no. 1-2, pp. 52–97, 2010.

[33] R. P. Stanley, "Hilbert functions of graded algebras," *Advances in Mathematics*, vol. 28, no. 1, pp. 57–83, 1978.

[34] I. Biswas, D. Gaiotto, S. Lahiri, and S. Minwalla, "Supersymmetric states of $N = 4$ Yang-Mills from giant gravitons," *Journal of High Energy Physics*, vol. 2007, no. 12, article 006, 2007.

[35] C. E. Beasley, "BPS branes from baryons," *Journal of High Energy Physics*, vol. 6, no. 11, article 015, pp. 297–319, 2002.

[36] D. Martelli, J. Sparks, and S.-T. Yau, "Sasaki-Einstein manifolds and volume minimisation," *Communications in Mathematical Physics*, vol. 280, no. 3, pp. 611–673, 2008.

[37] J. Distler, S. Mukhi, C. Papageorgakis, and M. Van Raamsdonk, "M2-branes on Mfolds," *Journal of High Energy Physics*, vol. 2008, no. 5, article 038, 2008.

[38] N. Lambert and D. Tong, "Membranes on an orbifold," *Physical Review Letters*, vol. 101, no. 4, Article ID 041602, 2008.

[39] K. Ueda and M. Yamazaki, "Toric Calabi-Yau four-folds dual to Chern-Simons-matter theories," *Journal of High Energy Physics*, vol. 2008, no. 12, article 045, 2008.

[40] Y. Imamura and K. Kimura, "Quiver Chern-Simons theories and crystals," *Journal of High Energy Physics*, vol. 2008, no. 10, article 114, 2008.

[41] S. Franco, A. Hanany, J. Park, and D. Rodriguez-Gomez, “Towards M2-brane theories for generic toric singularities,” *Journal of High Energy Physics*, vol. 2008, no. 12, article 110, 2008.

[42] A. Hanany and Y. H. He, “M2-branes and quiver Chern-Simons: a taxonomic study,” http://arxiv.org/abs/0811.4044.

[43] J. Davey, A. Hanany, N. Mekareeya, and G. Torri, “Phases of M2-brane theories,” *Journal of High Energy Physics*, vol. 2009, no. 6, article 025, 2009.

[44] A. Amariti, D. Forcella, L. Girardello, and A. Mariotti, “3D Seiberg-like dualities and M2 branes,” *Journal of High Energy Physics*, vol. 2010, no. 5, pp. 1–33, 2010.

[45] S. Franco, I. R. Klebanov, and D. Rodríguez-Gómez, “M2-branes on orbifolds of the cone over $Q^{1,1,1}$,” *Journal of High Energy Physics*, vol. 2009, no. 8, article 033, 2009.

[46] A. Hanany and Y. H. He, “Chern-Simons: Fano and Calabi-Yau,” http://arxiv.org/abs/0904.1847.

[47] J. Davey, A. Hanany, and J. Pasukonis, “On the classification of brane tilings,” http://arxiv.org/abs/0909.2868.

[48] J. Davey, A. Hanany, N. Mekareeya, and G. Torri, “Higgsing M2-brane theories,” *Journal of High Energy Physics*, vol. 2009, no. 11, article 028, 2009.

[49] D. Martelli and J. Sparks, “$\mathrm{AdS}_4/\mathrm{CFT}_3$ duals from M2-branes at hypersurface singularities and their deformations,” *Journal of High Energy Physics*, vol. 2009, no. 12, 2009.

[50] J. Davey, A. Hanany, N. Mekareeya, and G. Torri, “Brane tilings, M2-branes and Chern-Simons theories,” *Acta Physica Polonica B*, vol. 2, no. 3, pp. 639–655, 2009.

[51] J. Davey, A. Hanany, and R. K. Seong, “Counting orbifolds,” *Journal of High Energy Physics*, vol. 2010, no. 6, pp. 1–59, 2010.

[52] A. Hanany, N. Mekareeya, and A. Zaffaroni, “Partition functions for membrane theories,” *Journal of High Energy Physics*, vol. 2008, no. 9, article 090, 2008.

[53] B. E. W. Nilsson and C. N. Pope, “Hopf fibration of eleven-dimensional supergravity,” *Classical and Quantum Gravity*, vol. 1, no. 5, pp. 499–515, 1984.

[54] D. Martelli and J. Sparks, “Notes on toric Sasaki-Einstein seven-manifolds and $\mathrm{AdS}_4/\mathrm{CFT}_3$,” *Journal of High Energy Physics*, vol. 2008, no. 11, article 016, 2008.

[55] D. Fabbri, P. Fré, L. Gualtieri et al., “3D superconformal theories from Sasakian seven-manifolds: new non-trivial evidences for AdS_4/CFT_3,” *Nuclear Physics B*, vol. 577, no. 3, pp. 547–608, 2000.