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Research Article

A Global Approach with Cutoff Exponential Function, Mathematically Well Defined at the Outset, for Calculating the Casimir Energy: The Example of Scalar Field

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A global approach with cutoff exponential functions is used to obtain the Casimir energy of a massless scalar field in the presence of a spherical shell. The proposed method, mathematically well defined at the outset, makes use of two regulators, one of them to make the sum of the orders of Bessel functions finite and the other to regularize the integral involving the zeros of Bessel function. This procedure ensures a consistent mathematical handling in the calculations of the Casimir energy and allows a major comprehension on the regularization process when nontrivial symmetries are under consideration. In particular, we determine the Casimir energy of a scalar field, showing all kinds of divergences. We consider separately the contributions of the inner and outer regions of a spherical shell and show that the results obtained are in agreement with those known in the literature, and this gives a confirmation for the consistence of the proposed approach. The choice of the scalar field was due to its simplicity in terms of physical quantity spin.

1. Introduction

The relevance of the Casimir effect has increased over the decades since the seminal paper (1948) [1] by the Dutch Physicist Hendrik Casimir. This effect concerns to the appearance of an attractive force between two plates when they are placed close to each other. Casimir was the first to predict and explain the effect as a change in vacuum quantum fluctuations of the electromagnetic field.

Nowadays, the Casimir effect has been applied to a variety of quantum fields and geometries and it has gained a wider understanding as the effect which comes from the fluctuations of the zero point energy of a relativistic quantum field due to changes in its base manifold. This interpretation can be confirmed when we see the large range where the Casimir effect has been applied: the study of gauge fields with BRS symmetry [2], in the Higgs fields [3], in supersymmetric fields [4], in supergravity theory [5, 6], in superstrings [7], in the Maxwell-Chern-Simons fields [8], in relativistic strings [9], in M-theory [10], in cosmology [11], and in noncommutative spacetimes [12], among other subjects in the literature [13, 14], the review articles [15–17], and textbooks [18–23].

In this present work, the meaning of base manifold is that the confinement that the field is subjected is due to the presence of a sphere, where the boundary conditions take place. The point we aim to emphasize is that once the calculation of the Casimir effect involves dealing with infinite quantities, we need to use a regularization procedure appropriately defined. Many different regularization methods have been proposed and we can quote some of them: the summation mode method—using the general cutoff function [1], exponential cutoff function [24, 25], Green function [26–30], Green function through multiple scattering [31], exponential function and cutoff parameter [32, 33], zeta function [34–40], Abel-Plana formula [21, 41], or point-splitting [42–44]; the Green function method—using the point-splitting [45–48], Schwinger's source theory [49–51], or zeta function [52]; the statistical approach method—using the path integral formalism [53], or Green function [54]; as some examples among others. These methods are distinguished by the approach used to carry out the calculations of the Casimir energy, and it is clear that the physical result must be independent from the regulators or the method employed for them. But the literature has shown that the results found there exhibit a divergence among them.

In a general way, the methods used to obtain the Casimir effect lie on one of the two categories: a local procedure or a global one. With a local procedure, we mean one that the expression for the change of the vacuum energy is explicitly dependent on the variables of the base manifold, and only in the final step of calculations the integration over these variables is carried out. On the other hand, in a global one, we start with an expression for the vacuum energy where there is no space-time variables present as they already were integrated.

In the present work, we pretend detailing a global approach [55, 56] for the calculation of Casimir energy. In this method, mathematically well defined at the outset, we propose the use of two regulators into the cutoff exponential function, and we demonstrate that this regularization approach is one appropriate for the calculation of Casimir effect in the case of nontrivial symmetries, in particular a spherical symmetry.

With the use of scalar field, we can avoid the inherent complications brought by the vector nature of the electromagnetic field, and due to its simple structure, the scalar field usually becomes an effective tool to be used in the investigation of field proprieties as in these examples: in the dynamical Casimir effect [57], in the Casimir effect at finite temperature [58, 59], and in the Casimir effect on a presence of a gravitational field [60, 61], among others [62–64].

The paper is organized as follows: we detail in Section 2 the method to be used and why we need two cutoff parameters to obtain an regularized expression for the Casimir energy, which is the starting point for a consistent mathematical handling. Section 3 exhibits the calculations for the contributions of the inner and the outer regions of the spherical shell. We analyze in Section 4 the results and compare them with those ones in the literature and make some considerations.

2. The Global Procedure Proposed with Two Parameters

The starting point is the expression for the Casimir energy defined as the difference between the vacuum energy under a given boundary condition and the reference vacuum energy. When we consider a scalar field in the presence of a spherical shell, this vacuum energy is

$$E_0 = \sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\tau}\frac{1}{2}\hbar\omega_{jn}^{\tau}, \tag{2.1}$$

where ω_{jn}^{τ} are the mode frequencies. They are obtained when the boundary conditions are imposed on the field. In the absence of boundary conditions, the frequencies take some values which let us designate as $\omega_{jn}^{\tau(\mathrm{ref})}$ and these lead to the vacuum reference energy

$$E^{(\mathrm{ref})} = \sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\tau}\frac{1}{2}\hbar\omega_{jn}^{\tau(\mathrm{ref})}, \tag{2.2}$$

so the Casimir energy is $\mathcal{E} = E_0 - E^{(\mathrm{ref})}$. The boundary conditions due to a spherical shell with radius a are

$$\begin{aligned} kaj_j(ka) &= 0, \quad \text{for } r = a - 0, \\ A_j kaj_j(ka) + B_j kan_j(ka) &= 0, \quad \text{for } r = a + 0. \end{aligned} \tag{2.3}$$

The Casimir energy will be calculated by using the mode summation and the argument theorem (also known as argument principle [65–67]). This theorem gives the summation of zeros and poles of an analytic function as a contour integral. This contour is a curve that encompasses the interior region of the complex plane which contains the zeros and poles [65–67]. In our case, we are interested in the root functions which match the conditions (2.3). So, the following equations are appropriate as root functions

$$\begin{aligned} f_j^1(az) &= azj_j(az), \\ f_j^2(az) &= \cos\delta_j(z)\left[azj_j(az) + \tan\delta_j(z)azn_j(az)\right], \end{aligned} \tag{2.4}$$

where

$$z = k\left(=\frac{\omega}{c}\right), \qquad \delta_j(z) = zR - \frac{j\pi}{2}. \tag{2.5}$$

When we apply the argument theorem and carry out some handling, we get

$$\sum_{n=1}^{g}\omega_{jn}^{\tau} = \frac{c}{2\pi i}\oint_C dz\, z\frac{d}{dz}\log f_j^{\tau}(az). \tag{2.6}$$

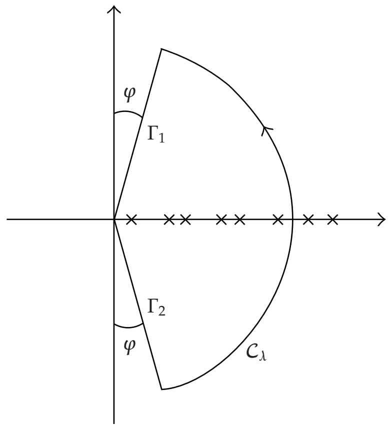

Figure 1: The path of integration in the complex plane.

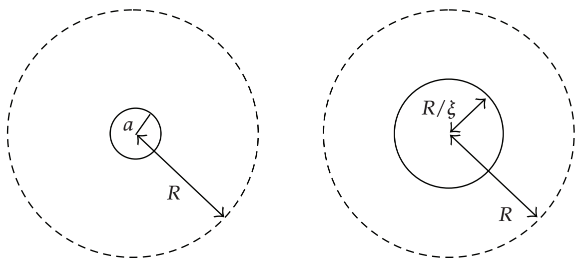

Figure 2: A sketch of the subtraction process which takes place on the regularization of the Casimir effect of a spherical shell.

In the above equation, the argument for logarithm must involve the product of all root functions. The contour to be taken on the calculations is given by [68] according to Figure 1.

The subtraction process (renormalization), defined by $\mathcal{E}$, can be schematically represented as in Figure 2.

The vacuum energy (2.1) which takes into account the boundary conditions can be used to obtain the reference energy in (2.2). This is done when we take the limit for the radius a going to infinity. This procedure is sensible, but it already has been made clear by Boyer [69, 70]. After all, we obtain for the Casimir energy

$$\begin{aligned}
\mathcal{E} &= \sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{\tau=1}^{4}\frac{1}{2}\hbar\left(\omega_{jn}^{\tau}-\omega_{jn}^{\tau(\mathrm{ref})}\right) \\
&= \lim_{\sigma\to 0,\,\epsilon\to 0,\,R\to\infty,\,\xi\to 1}\frac{\hbar c}{2\pi i}\sum_{j=0}^{\infty}\nu\exp(-\epsilon\nu)\oint_{C}dz\,z\exp(-\sigma z) \\
&\quad\times\frac{d}{dz}\left\{\log\left[f_j^{(1)}(az)f_j^{(2)}(az)\right]-\log\left[f_j^{(1)(\mathrm{ref})}\left(\frac{R}{\xi}z\right)f_j^{(2)(\mathrm{ref})}\left(\frac{R}{\xi}z\right)\right]\right\},
\end{aligned} \tag{2.7}$$

where $\nu = j + 1/2$. We can see from above that two exponential functions were used, one of them is the function under the integral sign, $\exp(-\sigma z)$, $(\sigma > 0)$, that stems from the argument theorem and the other is the function $\exp(-\epsilon\nu)$, $(\epsilon > 0)$, under summation sign on $j = \nu - (1/2)$. Now, group together these two developments, and (2.7) may be rewritten as

$$\mathcal{E} = -\frac{\hbar c}{\pi}\Re\sum_{j=0}^{\infty}\nu^2\exp(-\epsilon\nu)\int_0^{\infty\exp(-i\varphi)} dz\exp(-i\sigma\nu z)z\frac{d}{dz}\{\log[I_\nu(\nu az)] + \log[K_\nu(\nu az)]\} - E^{(\mathrm{ref})}, \tag{2.8}$$

where the limits for R and ξ have been taken into account. The other limits will be taken in an appropriate moment after the cancelation of possible remaining divergences.

3. Casimir Effect of a Spherical Shell: The Case of a Scalar Field

We now rewrite (2.8) in a more appropriate way, so that the contributions can be separated by regions as $\mathcal{E} = \mathcal{E}_I + \mathcal{E}_O$, where

$$\begin{aligned}\mathcal{E}_I = &-\frac{\hbar c}{\pi}\Re\left(\frac{1}{2}\right)^2\exp\left(-\epsilon\frac{1}{2}\right)\int_0^{\infty\exp(-i\varphi)} dz\, z\exp\left(-i\sigma\frac{1}{2}z\right)\frac{d}{dz}\left\{\log\left[I_{1/2}\left(\frac{1}{2}az\right)\right]\right\}\\ &-\frac{\hbar c}{\pi}\Re\sum_{j=1}^{\infty}\nu^2\exp(-\epsilon\nu)\int_0^{\infty\exp(-i\varphi)} dz\, z\exp(-i\sigma\nu z)\frac{d}{dz}\{\log[I_\nu(\nu az)]\} - E_I^{\mathrm{ref}}\end{aligned} \tag{3.1}$$

is the contribution due to the internal modes and

$$\begin{aligned}\mathcal{E}_O = &-\frac{\hbar c}{\pi}\Re\left(\frac{1}{2}\right)^2\exp\left(-\epsilon\frac{1}{2}\right)\int_0^{\infty\exp(-i\varphi)} dz\, z\exp\left(-i\sigma\frac{1}{2}z\right)\frac{d}{dz}\left\{\log\left[K_{1/2}\left(\frac{1}{2}az\right)\right]\right\}\\ &-\frac{\hbar c}{\pi}\Re\sum_{j=1}^{\infty}\nu^2\exp(-\epsilon\nu)\int_0^{\infty\exp(-i\varphi)} dz\, z\exp(-i\sigma\nu z)\frac{d}{dz}\{\log[K_\nu(\nu az)]\} - E_O^{\mathrm{ref}}\end{aligned} \tag{3.2}$$

is the contribution due to the external modes. As it can be observed, the above contributions were written in such a way that the term for $j = 0$ was detached from the summation on j. This has been done to the effect of making explicit the term on which we will focus attention as well as taking into account some developments already accomplished [55].

3.1. Internal Mode

Now, we proceed with the calculations of (3.1), and the first step is to take the Debye expansion for the Bessel functions up to order $\mathcal{O}(\nu^{-4})$ [71, 72]. The Debye expansion gives accurate results when we consider large order of $\nu = j + 1/2$ and larger arguments and that also makes an analytical treatment possible for the resulting expressions. So, we have

$$\mathcal{E}_I = E_I - E_I^{(\mathrm{ref})}, \tag{3.3}$$

where $E_I = E_{I,0} + E_{I,1} + E_{I,2} + E_{I,3} + E_{I,4}$ and the terms $E_{I,n}$ are given by

$$\begin{aligned} E_{I,0} = &-\frac{\hbar c}{2\pi}\Re \exp\left(-\epsilon\frac{1}{2}\right)\exp(-i\varphi)\int_0^\infty d\rho \exp(-i\sigma\rho\exp(-i\varphi)) \\ &\times\left[-\frac{1}{2} + \exp(-i\varphi)a\rho\coth(a\rho\ \exp(-i\varphi))\right], \end{aligned} \tag{3.4}$$

$$E_{I,1} = \frac{\hbar c}{\pi a}\sum_{j=1}^{\infty}\nu^2\int_0^\infty d\rho\left\{\log[\mathcal{O}_I(\nu,\rho)] - \sum_{k=1}^{4}\frac{\mathcal{U}_{(I,k)}(t)}{\nu^k}\right\}, \tag{3.5}$$

$$E_{I,2} = -\frac{\hbar c}{\pi}\Re\sum_{k=1}^{4}\sum_{j=1}^{\infty}\nu^{2-k}\exp(-\epsilon\nu)\int_0^{\infty\exp(-i\varphi)} dz\ \exp(-i\sigma\nu z) z\frac{d}{dz}\mathcal{U}_{(I,k)}(t), \tag{3.6}$$

$$E_{I,3} = \frac{\hbar c}{2\pi}\Re\sum_{j=1}^{\infty}\nu^2\exp(-\epsilon\nu)\int_0^{\infty\exp(-i\varphi)} dz\ \exp(-i\sigma\nu z)\frac{a^2z^2}{1+a^2z^2}, \tag{3.7}$$

$$E_{I,4} = -\frac{\hbar c}{\pi}\Re\sum_{j=1}^{\infty}\nu^3\exp(-\epsilon\nu)\int_0^{\infty\exp(-i\varphi)} dz\ \exp(-i\sigma\nu z)\sqrt{1+a^2z^2}, \tag{3.8}$$

with the definitions [37]

$$\mathcal{O}_I(\nu,\rho) = \sqrt{2\pi\nu}\left(1+\rho^2\right)^{1/4}\exp(-\nu\eta)I_\nu(\nu\rho), \tag{3.9}$$

$$\mathcal{U}_{(I,1)}(t) = \frac{t}{8} - \frac{5t^3}{24}, \tag{3.10}$$

$$\mathcal{U}_{(I,2)}(t) = \frac{t^2}{16} - \frac{3t^4}{8} + \frac{5t^6}{16}, \tag{3.11}$$

$$\mathcal{U}_{(I,3)}(t) = \frac{25t^3}{384} - \frac{531t^5}{640} + \frac{221t^7}{128} - \frac{1105t^9}{1152}, \tag{3.12}$$

$$\mathcal{U}_{(I,4)}(t) = \frac{13t^4}{128} - \frac{71t^6}{32} + \frac{531t^8}{64} - \frac{339t^{10}}{32} + \frac{565t^{12}}{128}. \tag{3.13}$$

The contributions (3.7) and (3.8) compound the zero-order terms of the Debye expansion. The contribution (3.5) stems from small values of the angular momentum j, and its value was already determined by [37]

$$E_{I,1} = 0.00024\frac{\hbar c}{\pi a}. \tag{3.14}$$

The contributions (3.6) to (3.8) are calculated taking into account the Euler-Maclaurin formula with remainder [73–75], and these were calculated by [55]

$$E_{I,2} = \frac{\hbar c}{\pi}\Re\left\{\frac{8099}{63839}\frac{1}{a} + \frac{7}{24}\frac{a}{\sigma^2} + \frac{11}{192}\frac{1}{a}\log\left(\frac{\sigma}{a}\right) + \frac{229}{40320}\frac{1}{a}\log(\epsilon) + i\frac{7801}{86684}\frac{1}{a}\right\}, \tag{3.15}$$

$$E_{I,3} = \frac{\hbar c}{\pi}\Re\left\{\frac{52529}{267528}\frac{1}{a} + i\left[-\frac{1}{3}\frac{1}{\sigma} - \frac{3}{4}\frac{1}{\sigma\epsilon} - \frac{1}{2}\frac{1}{\sigma\epsilon^2}\right]\right\}, \tag{3.16}$$

$$E_{I,4} = \frac{\hbar c}{\pi}\Re\left\{\frac{7375}{85696}\frac{1}{a} - \frac{11}{24}\frac{a}{\sigma^2} + 2\frac{a^3}{\sigma^4} - \frac{127}{1920}\frac{1}{a}\log\left(\frac{\sigma}{a}\right) - i\frac{2197}{21145}\frac{1}{a}\right\}. \tag{3.17}$$

Collecting the terms (3.14), (3.15), (3.16), and (3.17), we get

$$E_{I_{\text{partial}}} = 0.4095155894\frac{\hbar c}{\pi a} + \frac{\hbar c}{\pi}\left[-\frac{1}{6}\frac{a}{\sigma^2} + 2\frac{a^3}{\sigma^4} - \frac{17}{1920}\frac{1}{a}\log\left(\frac{\sigma}{a}\right) + \frac{229}{40320}\frac{1}{a}\log(\epsilon)\right]. \tag{3.18}$$

For (3.4), corresponding to $j = 0$, we obtain

$$E_{I,0} = \frac{\hbar c}{\pi}\left[-\frac{1}{24}\frac{\pi^2}{a} + \frac{1}{2}\frac{a}{\sigma^2}\right]. \tag{3.19}$$

So, the energy of a scalar field considering a spherical configuration due the internal modes is

$$\begin{aligned} E_I &= E_{I_{\text{partial}}} + E_{I,0} \\ &= -\frac{\hbar c}{\pi a}0.0017179275 + \frac{\hbar c}{\pi}\left[\frac{1}{3}\frac{a}{\sigma^2} + 2\frac{a^3}{\sigma^4} - \frac{17}{1920}\frac{1}{a}\log\left(\frac{\sigma}{a}\right) + \frac{229}{40320}\frac{1}{a}\log(\epsilon)\right]. \end{aligned} \tag{3.20}$$

The expression (3.20) shows in an undoubted way the need for a second regularized exponential function, $\exp(-\epsilon\nu)$, to make a consistent mathematical handling of the divergences possible. Both divergences, the logarithm in (3.15) and the polynomial in (3.16), stem from the summation on j. This type of divergence was already observed in [76], but only with the procedure established here this discard turns to be completely justified as an appropriate regularization allows the real part of (3.16) to be taken in an unambiguous way.

3.2. External Mode

The contribution of the external modes comes by (3.2). We proceed with the calculations in an analogous way to that of the previous subsection. So,

$$\mathcal{E}_O = E_O - E_O^{(\text{ref})}, \tag{3.21}$$

where $E_O = E_{O,0} + E_{O,1} + E_{O,2} + E_{O,3} + E_{O,4}$ and

$$E_{O,0} = -\frac{\hbar c}{2\pi}\Re \exp\left(-\epsilon\frac{1}{2}\right)\exp(-i\varphi)\int_0^\infty d\rho \exp(-i\sigma\rho\exp(-i\varphi))\left[-\frac{1}{2} - \exp(-i\varphi)a\rho\right], \tag{3.22}$$

$$E_{O,1} = \frac{\hbar c}{\pi a}\sum_{j=1}^{\infty}\nu^2\int_0^\infty d\rho\left\{\log[\mathcal{K}_O(\nu,\rho)] - \sum_{k=1}^{4}\frac{\mathcal{U}_{(O,k)}(t)}{\nu^k}\right\}, \tag{3.23}$$

$$E_{0,2} = -\frac{\hbar c}{\pi}\Re\sum_{k=1}^{4}\sum_{j=1}^{\infty}\nu^{2-k}\exp(-\epsilon\nu)\int_0^{\infty\exp(-i\varphi)} dz \exp(-i\sigma\nu z) z\frac{d}{dz}\mathcal{U}_{(O,k)}(t), \tag{3.24}$$

$$E_{O,3} = \frac{\hbar c}{2\pi}\Re\sum_{j=1}^{\infty}\nu^2\exp(-\epsilon\nu)\int_0^{\infty\exp(-i\varphi)} dz\exp(-i\sigma\nu z)\frac{a^2z^2}{1+a^2z^2}, \tag{3.25}$$

$$E_{O,4} = \frac{\hbar c}{\pi}\Re\sum_{j=1}^{\infty}\nu^3\exp(-\epsilon\nu)\int_0^{\infty\exp(-i\varphi)} dz\exp(-i\sigma\nu z)\sqrt{1+a^2z^2}, \tag{3.26}$$

with the following definitions [37]:

$$\mathcal{K}_O(\nu,\rho) = \sqrt{\frac{2\nu}{\pi}}\left(1+\rho^2\right)^{1/4}\exp(\nu\eta)K_\nu(\nu\rho), \tag{3.27}$$

$$\mathcal{U}_{(O,1)}(t) = -\mathcal{U}_{(I,1)}(t), \tag{3.28}$$

$$\mathcal{U}_{(O,2)}(t) = \mathcal{U}_{(I,2)}(t), \tag{3.29}$$

$$\mathcal{U}_{(O,3)}(t) = -\mathcal{U}_{(I,3)}(t), \tag{3.30}$$

$$\mathcal{U}_{(O,4)}(t) = \mathcal{U}_{(I,4)}(t), \tag{3.31}$$

where the $\mathcal{U}_{(O,k)}$ are given by (3.10) to (3.13), respectively. The term (3.28) was numerically determined by [37]

$$E_{O,1} = -0.00054\frac{\hbar c}{\pi a}. \tag{3.32}$$

The other contributions are calculated following the analogous procedure detailed in the previous subsection:

$$\begin{aligned} E_{O,2} &= \frac{\hbar c}{\pi}\Re\left\{-\frac{5821}{56688}\frac{1}{a} - \frac{7}{24}\frac{a}{\sigma^2} - \frac{11}{192}\frac{1}{a}\log\left(\frac{\sigma}{a}\right) - \frac{229}{40320}\frac{1}{a}\log(\epsilon) - i\frac{7801}{86684}\frac{1}{a}\right\}, \\ E_{O,3} &= \mathcal{E}_{I,3}, \\ E_{O,4} &= -\mathcal{E}_{I,4}\ . \end{aligned} \tag{3.33}$$

After collecting the terms (3.32) and (3.33), we have

$$E_{O_{\text{partial}}} = 0.01056399145\frac{\hbar c}{\pi a} + \frac{\hbar c}{\pi}\left[\frac{1}{6}\frac{a}{\sigma^2} - 2\frac{a^3}{\sigma^4} + \frac{17}{1920}\frac{1}{a}\log\left(\frac{\sigma}{a}\right) - \frac{229}{40320}\frac{1}{a}\log(\epsilon)\right]. \tag{3.34}$$

The contribution (3.22), related to $j = 0$, when we repeat the calculation gives

$$E_{O,0} = \frac{\hbar c}{\pi}\left[-\frac{1}{2}\frac{a}{\sigma^2}\right]. \tag{3.35}$$

Gathering together (3.34) and (3.35), we get the total contribution to the energy of a scalar field of a spherical configuration due the external modes

$$\begin{aligned} E_O &= E_{O_{\text{partial}}} + E_{O,0} \\ &= \frac{\hbar c}{\pi a}0.01056399145 + \frac{\hbar c}{\pi}\left[-\frac{1}{3}\frac{a}{\sigma^2} - 2\frac{a^3}{\sigma^4} + \frac{17}{1920}\frac{1}{a}\log\left(\frac{\sigma}{a}\right) - \frac{229}{40320}\frac{1}{a}\log(\epsilon)\right]. \end{aligned} \tag{3.36}$$

Our next task is to determine the reference energy and take the regularizations as indicated by (3.3) and (3.21).

4. The Regularized Results

Calculating the reference energy, we get

$$E_{\pm}^{(\text{ref})} = \pm\frac{R}{\xi}\frac{f(\epsilon)}{\sigma^2}, \tag{4.1}$$

where the plus sign refers to $\mathcal{E}_I$ and the minus sign to $\mathcal{E}_O$ and

$$f(\epsilon) = \exp\left(-\frac{\epsilon}{2}\right)(3\exp(\epsilon) - 1)(\exp(\epsilon) - 1)^{-2}. \tag{4.2}$$

Now, we can gather together the internal (3.20) and external (3.36) contributions, taking into account (4.1), to obtain the Casimir effect for a scalar field due to the presence of a spherical shell with radius a

$$\mathcal{E}(a) = \frac{\hbar c}{a}0.002815789609. \tag{4.3}$$

This result is free of divergences since we get an exact cancelation for the terms which depend on the cutoff parameters. Equation (4.3) is in agreement with that obtained by [37], through the zeta function method, and with that in [77], which uses the Green function formalism and the dimensional analytical extension (in this reference the starting point is the expression for the force).

5. Conclusions

Our purpose in this work was to show the form and nature of each divergent term that appears in the calculation of Casimir energy and demonstrate that the method proposed is mathematically consistent and that it is in accordance with the results existing in literature.

To this end, we show explicitly, to the scalar field, the characteristic of the divergent terms calculated in (3.20), (3.36), and (4.1) as a function of the geometrical proprieties of boundary if we rewrite the divergent part of those assuming a dimensionless parameter $\varepsilon = (\epsilon a)^{357/155}$, with $\dim(\epsilon) = L^{-1}$, so

$$\mathcal{E}_{\pm} = \pm\hbar c\left[\frac{3}{\pi^2}V(a)\frac{1}{\sigma^4} - \frac{17}{3840\pi^2}S(a)\kappa^3\log(\sigma\varepsilon) + \frac{1}{3\pi^2}S(a)\kappa\frac{1}{\sigma^2} - \frac{1}{4\pi^2}S\left(\frac{R}{\xi}\right)\kappa S\left(\frac{R}{\xi}\right)\frac{f(\varepsilon)}{\sigma^2}\right], \tag{5.1}$$

where the plus sign refers to the index I while the minus sign refers to the index O, and $\kappa = 1/a$ is the curvature. In (5.1), $V(a)$ is a volume, $S(a)$ is an area, and σ and ϵ are cutoff parameters. As we can see, the second and fourth terms in (5.1) explain why two regulators are required to get a well-defined expression for the Casimir energy of the scalar field. This is the same case when we consider an electromagnetic field (see [55]). The result (5.1) is in agreement with [78], except for the divergence due to $\log(\epsilon)$ and due to the relative self-energy of the spherical shell, $f(\varepsilon)/\sigma^2$, that does not appear there.

Our purpose in this work is to confirm that the prescription in (2.7) works well when we assume non trivial symmetries for the fields. In fact, the approach has succeed in demonstrating the cancelation of all types of divergences appearing in the expression for the Casimir energy of the scalar field. Besides, this calculation presented at this work shows the desired agreement with the results existing in the literature. Furthermore, as it has been mentioned by the authors in [79–81], a better understanding of the quantum field theory undoubtedly involves the necessity to understand these infinities.

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