Hindawi
Advances in Fuzzy Systems
Volume 2020, Article ID 8841670, 9 pages
https://doi.org/10.1155/2020/8841670

Research Article

μ-Fuzzy Filters in Distributive Lattices

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Received 24 July 2020; Accepted 30 November 2020; Published 29 December 2020

Academic Editor: Antonin Dvorák

In this paper, we introduce the concept of μ-fuzzy filters in distributive lattices. We study the special class of fuzzy filters called μ-fuzzy filters, which is isomorphic to the set of all fuzzy ideals of the lattice of coannihilators. We observe that every μ-fuzzy filter is the intersection of all prime μ-fuzzy filters containing it. We also topologize the set of all prime μ-fuzzy filters of a distributive lattice. Properties of the space are also studied. We show that there is a one-to-one correspondence between the class of μ-fuzzy filters and the lattice of all open sets in X_μ. It is proved that the space X_μ is a T_0 space.

1. Introduction

In 1970, Mandelker [1] introduced the concept of relative annihilators as a natural generalization of relative pseudocomplement and he characterized distributive lattices with the help of these annihilators. The concept of coannihilators and μ-filters in a distributive lattice with greatest element "1" was introduced by Rao and Badawy [2] and they characterized μ-filters in terms of coannihilators. For a filter F in L, $\mu(F) = \{(x)^{++}: x \in F\}$ is an ideal in the set $A^+(L)$ of all coannihilators, and conversely $\overleftarrow{\mu}(I) = \{x \in L: (x)^{++} \in I\}$ is a filter in L when I is any ideal in $A^+(L)$. A filter F of L is called a μ-filter if $\overleftarrow{\mu}\mu(F) = F$.

In 1965, Zadeh [3] mathematically formulated the fuzzy subset concept. He defined fuzzy subset of a nonempty set as a collection of objects with grade of membership in a continuum, with each object being assigned a value between 0 and 1 by a membership function. Fuzzy set theory was guided by the assumption that classical sets were not natural, appropriate, or useful notions in describing the real-life problems, because every object encountered in this real physical world carries some degree of fuzziness. A lot of work on fuzzy sets has come into being with many applications to various fields such as computer science, artificial intelligence, expert systems, control systems, decision-making, medical diagnosis, management science, operations research, pattern recognition, neural network, and others (see [4–7]).

In 1971, Rosenfeld used the notion of a fuzzy subset of a set to introduce the concept of a fuzzy subgroup of a group [8]. Rosenfeld's paper inspired the development of fuzzy abstract algebra. Since then, several authors have developed interesting results on fuzzy subgroups (see [9–17]), fuzzy ideals of rings (see [16, 18–21]), and fuzzy ideals of lattices (see [22–28]).

Alaba and Norahun [29] studied the concept of α-fuzzy ideals of a distributive lattice in terms of annulates. They also studied the space of prime α-fuzzy ideals of a distributive lattice. In this paper, we introduce the dual of the concept of α-fuzzy ideals which is called μ-fuzzy filters in a distributive lattice with greatest element "1." We study the special class of fuzzy filters called μ-fuzzy filters. We prove that the set of all μ-fuzzy filters of a distributive lattice forms a complete distributive lattice isomorphic to the set of all fuzzy ideals of $A^+(L)$. We also show that there is a one-to-one correspondence between the class of prime μ-fuzzy filters of L and the set of all prime ideals of $A^+(L)$. We prove that every μ-fuzzy filter is the intersection of all prime μ-fuzzy filters containing it. Moreover, we study the space of all prime μ-fuzzy filters in a distributive lattice. The set of prime μ-fuzzy filters of L is denoted by X_μ. For a μ-fuzzy filter θ of L, open subset of X_μ is of the form $X(\theta) = \{\eta \in X_\mu: \theta \nsubseteq \eta\}$

and $V(\theta) = \{\eta \in X_\mu\colon \theta \subseteq \eta\}$ is a closed set. We also show that the set of all open sets of the form $X(x_\beta) = \{\eta \in X_\mu\colon x_\beta \notin \eta,\ x \in L,\ \beta \in (0,1]\}$ forms a basis for the open sets of X_μ. The set of all μ-fuzzy filters of L is isomorphic with the set of all open sets in X_μ.

2. Preliminaries

We refer to Birkhoff [30] for the elementary properties of lattices.

Definition 1 (see [2]). For any set S of a lattice L, define S^+ as follows:

$$S^+ = \{x \in L\colon s \vee x = 1, \text{ for all } s \in S\}. \tag{1}$$

Here, S^+ is called the coannihilator of S. If $S = \{x\}$, we write $(x)^+$ instead of $(\{x\})^+$. Then, clearly $L^+ = \{1\}$ and $(1)^+ = L$. For any subset S of a lattice L, it is clear that S^+ is a filter in L.

Lemma 1 (see [2]). *For any $x, y \in L$, the following conditions hold.*

(1) $x \le y \Longrightarrow (x)^+ \subseteq (y)^+$
(2) $(x \wedge y)^+ = (x)^+ \cap (y)^+$
(3) $(x \vee y)^{++} = (x)^{++} \cap (y)^{++}$
(4) $(x)^+ = L$ if and only if $x = 1$

The set of all coannihilator denotes $A^+(L)$. Each coannihilator is a coannihilator filter, and hence, for two coannihilators $(x)^+$ and $(y)^+$, their supremum and infimum in $A^+(L)$ are

$$\begin{aligned}(a \vee b)^{++} &= (a)^{++} \cap (b)^{++},\\ (a)^{++} \,\underline{\vee}\, (b)^{++} &= (a \wedge b)^{++} a,\end{aligned} \tag{2}$$

respectively.

In a distributive lattice L with 1, the set of all coannihilators $A^+(L)$ of L is a lattice $(A^+(L), \cap, \vee)$ and a sublattice of the Boolean algebra of coannihilator filters of L.

For a filter F in L,

$$\mu(F) = \{(x)^{++}\colon x \in F\} \tag{3}$$

is an ideal in $A^+(L)$ and the set

$$\overleftarrow{\mu}(I) = \{x \in L\colon (x)^{++} \in I\} \tag{4}$$

is a filter of L when I is any ideal in $A^+(L)$. A filter F of L is called a μ-filter if $\overleftarrow{\mu}\mu(F) = F$.

Definition 2 (see [3]). Let X be any nonempty set. A mapping $\mu\colon X \longrightarrow [0,1]$ is called a fuzzy subset of X.

The unit interval $[0,1]$ together with the operations min and max form a complete lattice satisfying the infinite meet distributive law. We often write $\wedge$ for minimum or infimum and $\vee$ for maximum or supremum. That is, for all $\alpha, \beta \in [0,1]$, we have, $\alpha \wedge \beta = \min\{\alpha, \beta\}$ and $\alpha \vee \beta = \max\{\alpha, \beta\}$.

The characteristic function of any set A is defined as

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A,\\ 0, & \text{if } x \notin A. \end{cases} \tag{5}$$

Definition 3 (see [8]). Let μ and θ be fuzzy subsets of a set A. Define the fuzzy subsets $\mu \cup \theta$ and $\mu \cap \theta$ of A as follows: for each $x \in A$,

$$(\mu \cup \theta)(x) = \mu(x) \vee \theta(x) \text{ and } (\mu \cap \theta)(x) = \mu(x) \wedge \theta(x). \tag{6}$$

Then, $\mu \cup \theta$ and $\mu \cap \theta$ are called the union and intersection of μ and θ, respectively.

For any collection, $\{\mu_i\colon i \in I\}$ of fuzzy subsets of X, where I is a nonempty index set, and the least upper bound $\cup_{i\in I}\mu_i$ and the greatest lower bound $\cap_{i\in I}\mu_i$ of the μ_i's are given for each $x \in X$,

$$\left(\bigcup_{i\in I} \mu_i\right)(x) = \vee_{i\in I}\mu_i(x) \text{ and } \left(\bigcap_{i\in I} \mu_i\right)(x) = \wedge_{i\in I}\mu_i(x), \tag{7}$$

respectively.

For each $t \in [0,1]$, the set

$$\mu_t = \{x \in A\colon \mu(x) \ge t\} \tag{8}$$

is called the level subset of μ at t [3].

Definition 4 (see [27]). A fuzzy subset μ of a lattice L is called a fuzzy ideal of L if; for all $x, y \in L$, the following conditions are satisfied:

(1) $\mu(0) = 1$
(2) $\mu(x \vee y) \ge \mu(x) \wedge \mu(y)$
(3) $\mu(x \wedge y) \ge \mu(x) \vee \mu(y)$

Definition 5 (see [27]). A fuzzy subset μ of a lattice L is called a fuzzy filter of L if, for all $x, y \in L$, the following condition is satisfied:

(1) $\mu(1) = 1$
(2) $\mu(x \vee y) \ge \mu(x) \vee \mu(y)$
(3) $\mu(x \wedge y) \ge \mu(x) \wedge \mu(y)$

In [27], Swamy and Raju observed the following:

(1) A fuzzy subset μ of a lattice L is a fuzzy ideal of L if and only if

$$\mu(0) = 1 \text{ and } \mu(x \vee y) = \mu(x) \wedge \mu(y), \quad \text{for all } x, y \in L. \tag{9}$$

(2) A fuzzy subset μ of a lattice L is a fuzzy filter of L if and only if

$$\mu(1) = 1 \text{ and } \mu(x \wedge y) = \mu(x) \wedge \mu(y), \quad \text{for all } x, y \in L. \tag{10}$$

Let μ be a fuzzy subset of a lattice L. The smallest fuzzy filter of L containing μ is called a fuzzy filter of L induced by μ and denoted by $[\mu)$ and

$$[\mu) = \cap\{\theta \in FF(L):\ \mu \subseteq \theta\}. \quad (11)$$

Lemma 2 (see [23]). *For any two fuzzy subsets μ and θ of a distributive lattice L, we have*

$$(\mu \cdot \theta] = (\mu] \wedge (\theta]. \quad (12)$$

The above result works dually.

For any two fuzzy subsets μ and θ of a distributive lattice L, we have

$$[\mu + \theta) = [\mu) \wedge [\theta). \quad (13)$$

The binary operations "+" and "·" on the set of all fuzzy subsets of a distributive lattice L are as follows:

$$(\mu + \theta)(x) = \text{Sup}\{\mu(y) \wedge \theta(z):\ y, z \in L, y \vee z = x\} \text{ and } (\mu \cdot \theta)(x) = \text{Sup}\{\mu(y) \wedge \theta(z):\ y,\ z \in L, y \wedge z = x\}. \quad (14)$$

If μ and θ are fuzzy ideals of L, then $\mu \cdot \theta = \mu \wedge \theta = \mu \cap \theta$ and $\mu + \theta = \mu \vee \theta$

If μ and θ are fuzzy filters of L, then $\mu + \theta = \mu \wedge \theta$ and $\mu \cdot \theta = \mu \vee \theta$

The set of all fuzzy filters of L is denoted by $FF(L)$.

3. μ-Fuzzy Filters

In this section, we introduce the concept of μ-fuzzy filters in a distributive lattice with greatest element "1." We study some basic properties of the class of μ-fuzzy filters. We prove that the class of μ-fuzzy filters forms a complete distributive lattice isomorphic to the class of fuzzy filters of $A^+(L)$. We also show that there is a one-to-one correspondence between the set of all prime μ-fuzzy filters of L and prime fuzzy ideals of $A^+(L)$. Finally, we observe that every μ-fuzzy filter is the intersection of all prime μ-fuzzy filters containing it.

Throughout the rest of this paper, L stands for the distributive lattice with greatest element "1" unless otherwise mentioned.

Theorem 1. *Let θ be a fuzzy filter of L. Then, the fuzzy subset $\mu(\theta)$ of $A^+(L)$ defined by*

$$\mu(\theta)((x)^{++}) = \text{Sup}\{\theta(y):\ (y)^{++} = (x)^{++},\quad y \in L\} \quad (15)$$

is a fuzzy ideal of $A^+(L)$.

Proof. Let θ be a fuzzy filter of L. Clearly, $\mu(\theta)((1)^{++}) = 1$. For any $(x)^{++}, (y)^{++} \in A^+(L)$,

$$\begin{aligned}\mu(\theta)((x)^{++}) \wedge \mu(\theta)((y)^{++}) &= \text{Sup}\{\theta(a):\ (a)^{++} = (x)^{++}, a \in L\} \\ &\quad \wedge \text{Sup}\{\theta(b):\ (b)^{++} = (y)^{++}, b \in L\} \\ &= \text{Sup}\{\theta(a) \wedge \theta(b):\ (a)^{++} = (x)^{++}, (b)^{++} = (y)^{++}\} \\ &\leq \text{Sup}\{\theta(a) \wedge \theta(b):\ (a)^{++} \underline{\vee}\ (b)^{++} = (x)^{++} \underline{\vee}\ (y)^{++}\} \\ &= \text{Sup}\{\theta(a \wedge b):\ (a \wedge b)^{++} = (x \wedge y)^{++}\} \\ &\leq \text{Sup}\{\theta(c):\ (c)^{++} = (x \wedge y)^{++}\} = \mu(\theta)((x)^{++} \underline{\vee}\ (y)^{++}).\end{aligned} \quad (16)$$

Thus, $\mu(\theta)((x)^{++} \vee (y)^{++}) \geq \mu(\theta)((x)^{++}) \wedge \mu(\theta)((y)^{++})$.

On the other hand,

$$\begin{aligned}\mu(\theta)((x)^{++}) &= \text{Sup}\{\theta(a):\ (a)^{++} = (x)^{++}\} \\ &\leq \text{Sup}\{\theta(a \vee y):\ (a)^{++} \wedge (y)^{++} = (x)^{++} \wedge (y)^{++}\} \\ &\leq \text{Sup}\{\theta(c):\ (c)^{++} = (x \vee y)^{++}\} \\ &= \mu(\theta)((x)^{++} \wedge (y)^{++}).\end{aligned} \quad (17)$$

Similarly, $\mu(\theta)((y)^{++}) \leq \mu(\theta)((x)^{++} \wedge (y)^{++})$. So,

$$\mu(\theta)((x)^{++} \wedge (y)^{++}) \geq \mu(\theta)((x)^{++}) \vee \mu(\theta)((y)^{++}). \quad (18)$$

Hence, $\mu(\theta)$ is a fuzzy ideal of $A^+(L)$.

Lemma 3. *Let η be a fuzzy ideal of $A^+(L)$. Then, the fuzzy subset $\overleftarrow{\mu}(\eta)$ of L defined as $\overleftarrow{\mu}(\eta)(x) = \eta((x)^{++})$ is a fuzzy filter of L.*

Proof. Let η be a fuzzy ideal of $A^+(L)$. Since $(1)^{++}$ is the smallest element of $A^+(L)$, we get $\overleftarrow{\mu}(\eta)(1) = 1$. For any $x, y \in L$,

$$\overleftarrow{\mu}(\eta)(x \vee y) = \eta((x)^{++}) \wedge \eta((y)^{++}) = \overleftarrow{\mu}(\eta)(x) \wedge \overleftarrow{\mu}(\eta)(y). \quad (19)$$

Thus, $\overleftarrow{\mu}(\eta)$ is a fuzzy filter of L.

Lemma 4. *If θ and η are fuzzy filters of L, then $\theta \subseteq \eta$ implies $\mu(\theta) \subseteq \mu(\eta)$.*

Lemma 5. *If θ, η are fuzzy ideals of $A^+(L)$, then $\theta \subseteq \eta$ implies $\overleftarrow{\mu}(\theta) \subseteq \overleftarrow{\mu}(\eta)$.*

Theorem 2. *The set $FI(A^+(L))$ of all fuzzy ideals of $A^+(L)$ forms a complete distributive lattice, where the infimum and supremum of any family $\{\theta_j:\ j \in J\}$ of fuzzy ideals are given by*

$$\wedge \eta_j = \cap \eta_j \text{ and } \vee \eta_j = \left[\bigcup \eta_j\right). \quad (20)$$

Theorem 3. *The mapping μ is a homomorphism of $FF(L)$ into $FI(A^+(L))$.*

Proof. Let η and θ be two fuzzy filters of L. Then, by Lemma 4, we have $\mu(\eta \cap \theta) \subseteq \mu(\eta) \cap \mu(\theta)$ and $\mu(\eta) \vee \mu(\theta) \subseteq \mu(\eta \vee \theta)$. For any $(x)^{++} \in A^+(L)$,

$$\mu(\eta)((x)^{++}) \wedge \mu(\theta)((y)^{++}) = \text{Sup}\{\eta(a): (a)^{++} = (x)^{++}\} \wedge \text{Sup}\{\theta(b): (b)^{++} = (y)^{++}\}. \quad (21)$$

Since $(a)^{++} = (x)^{++}$ and $(b)^{++} = (y)^{++}$, we get $(a \wedge b)^{++} = (x)^{++}$. Based on this, we have

$$\begin{aligned}\mu(\eta)((x)^{++}) \wedge \mu(\theta)((x)^{++}) &\leq \text{Sup}\{\eta(a \wedge b): (a \wedge b)^{++} = (x)^{++}\} \\ \wedge \text{Sup}\{\theta(a \wedge b): (a \wedge b)^{++} = (x)^{++}\} &= \text{Sup}\{\eta(a \wedge b) \wedge \theta(a \wedge b): (a \wedge b)^{++} = (x)^{++}\} \\ &= \text{Sup}\{(\eta \cap \theta)(a \wedge b): (a \wedge b)^{++} = (x)^{++}\} \leq \text{Sup}\{(\eta \cap \theta)(c): (c)^{++} = (x)^{++}\} \\ &= \mu(\eta \cap \theta)((x)^{++}).\end{aligned} \quad (22)$$

Thus, $\mu(\eta) \cap \mu(\theta) \subseteq \mu(\eta \cap \theta)$. So, $\mu(\eta) \cap \mu(\theta) = \mu(\eta \cap \theta)$.

On the other hand,

$$\begin{aligned}\mu(\eta \vee \theta)((x)^{++}) &= \text{Sup}\{(\eta \vee \theta)(a): (a)^{++} = (x)^{++}\} \\ &= \text{Sup}\{\text{Sup}\{\eta(y) \wedge \theta(z): a = y \wedge z\}, (y \wedge z)^{++} = (x)^{++}\} \\ &\leq \text{Sup}\{\text{Sup}\{\eta(b_1) \wedge \theta(b_2): (b_1)^{++} = (y)^{++}, (b_2)^{++} = (z)^{++}\}, (y \wedge z)^{++} = (x)^{++}\} \\ &= \text{Sup}\{\text{Sup}\{\eta(b_1): (b_1)^{++} = (y)^{++}\} \wedge \text{Sup}\{\theta(b_2): (b_2)^{++} = (z)^{++}\}, (y \wedge z)^{++} = (x)^{++}\} \\ &= \text{Sup}\{\mu(\eta)(y)^{++} \wedge \mu(\theta)(z)^{++}: (y \wedge z)^{++} = (x)^{++}\} \\ &= \text{Sup}\{\mu(\eta)(y)^{++} \wedge \mu(\theta)(z)^{++}: (y)^{++} \underline{\vee} (z)^{++} = (x)^{++}\} = (\mu(\eta) \underline{\vee} \mu(\theta))((x)^{++}).\end{aligned} \quad (23)$$

Then, $\mu(\eta \vee \theta) \subseteq \mu(\eta) \vee \mu(\theta)$. Thus, $\mu(\eta \vee \theta) = \mu(\eta) \vee \mu(\theta)$. So, μ is a homomorphism.

Corollary 1. *For any two fuzzy filters θ and η of L, we have*

$$\overleftarrow{\mu}\mu(\theta \cap \eta) = \overleftarrow{\mu}\mu(\theta) \cap \overleftarrow{\mu}\mu(\eta). \quad (24)$$

Proof. For any $x \in L$, $\overleftarrow{\mu}\mu(\eta \cap \theta)(x) = \mu(\eta \cap \theta)((x)^{++}))$. Since μ is a homomorphism, we have $\overleftarrow{\mu}\mu(\eta \cap \theta) = \overleftarrow{\mu}\mu(\eta) \cap \overleftarrow{\mu}\mu(\theta)$.

Lemma 6. *For any fuzzy ideal θ of $A^+(L)$, $\mu\overleftarrow{\mu}(\theta) = \theta$.*

Proof. Since θ is a fuzzy ideal of $A^+(L)$, by Lemma 3, $\overleftarrow{\mu}(\theta)$ is a fuzzy filter of L and $\mu\overleftarrow{\mu}(\theta)$ is a fuzzy ideal of $A^+(L)$. Now, we proceed to show $\mu\overleftarrow{\mu}(\theta) = \theta$.

$$\begin{aligned}\mu\overleftarrow{\mu}(\theta)((x)^{++}) &= \text{Sup}\left\{\overleftarrow{\mu}(\theta)(a): (a)^{++} = (x)^{++}\right\} \\ &= \text{Sup}\{\theta((a)^{++}): (a)^{++} = (x)^{++}\} = \theta((x)^{++}).\end{aligned} \quad (25)$$

Thus, $\mu\overleftarrow{\mu}(\theta) = \theta$.

Lemma 7. *For any fuzzy filter θ of L, the map $\theta \longrightarrow \overleftarrow{\mu}\mu(\theta)$ is a closure operator on $FF(L)$. That is,*

(1) $\theta \subseteq \overleftarrow{\mu}\mu(\theta)$
(2) $\overleftarrow{\mu}\mu(\overleftarrow{\mu}\mu t(\theta)) = \overleftarrow{\mu}\mu(\theta)$
(3) $\theta \subseteq \eta \Longrightarrow \overleftarrow{\mu}\mu(\theta) \subseteq \overleftarrow{\mu}\mu(\eta)$, for any two fuzzy filters θ, η of L

Now, we define μ-fuzzy filter.

Definition 6. A fuzzy filter θ of L is called a μ-fuzzy filter of L if $\theta = \overleftarrow{\mu}\mu(\theta)$.

Thus, μ-fuzzy filters are simply the closed elements with respect to the closure operator of Lemma 7, and $\overleftarrow{\mu}\mu(\theta)$ is the smallest μ-fuzzy filter containing θ, for any fuzzy filter θ of L.

Theorem 4. *For a nonempty fuzzy subset θ of L, θ is a μ-fuzzy filter if and only if each level subset of θ is a μ-filter of L.*

Proof. Let θ be a μ-fuzzy filter of L. Then, $\theta_t = (\overleftarrow{\mu}\mu t(\theta))_t$. Now, we proceed to show $\overleftarrow{\mu}\mu(\theta_t) = (\overleftarrow{\mu}\mu t(\theta))_t$ for all $t \in [0, 1]$. We know that $(\overleftarrow{\mu}\mu t(\theta))_t \subseteq \overleftarrow{\mu}\mu(\theta_t)$. To show the other inclusion, let $x \in \overleftarrow{\mu}\mu(\theta_t)$. Then, $(x)^{++} \in \mu(\theta_t)$, and there is $y \in \theta_t$ such that $(x)^{++} = (y)^{++}$. Thus, $\overleftarrow{\mu}\mu(\theta)(x) \geq t$. So, $\theta_t = \overleftarrow{\mu}\mu(\theta_t)$. Therefore, each level subset of θ is a μ-filter of L.

Conversely, assume that each level subset of θ is a μ-filter. Then, θ is a fuzzy filter and $\theta \subseteq \overleftarrow{\mu}\mu(\theta)$. To prove our claim, let $t = \overleftarrow{\mu}\mu(\theta)(x)$. Then, for each $\epsilon > 0$, there is $a \in L$ such that

$(a)^{++} = (x)^{++}$ and $\theta(a) > t - \epsilon$, which implies $a \in \theta_{t-\epsilon}$, $(a)^{++} = (x)^{++}$ and $x \in \overleftarrow{\mu}\mu(\theta_{t-\epsilon}) = \theta_{t-\epsilon}$. This shows that $x \in \cap_{\epsilon>0}\theta_{t-\epsilon} = \theta_t$. Thus, $\overrightarrow{\mu}\mu(\theta) \subseteq \theta$. So, θ is a μ-fuzzy filter of L.

Corollary 2. *For a nonempty subset F of L, F is a μ-filter if and only if χ_F is a μ-fuzzy filter of L.*

Theorem 5. *Let θ be a fuzzy filter of L. Then, θ is a μ-fuzzy filter if and only if, for each $x, y \in L$, $(x)^+ = (y)^+$ implies $\theta(x) = \theta(y)$.*

Proof. Let θ be a μ-fuzzy filter of L and $x, y \in L$ such that $(x)^{++} = (y)^{++}$. Then,

$$\begin{aligned}\theta(x) &= \operatorname{Sup}\{\theta(a)\colon (a)^{++} = (x)^{++}, a \in L\} \\ &= \operatorname{Sup}\{\theta(a)\colon (a)^{++} = (y)^{++}, a \in L\} = \overleftarrow{\mu}\mu(\theta)(y) = \theta(y).\end{aligned} \tag{26}$$

Conversely, suppose that for each $x, y \in L$, $(x)^{++} = (y)^{++}$ implies $\theta(x) = \theta(y)$. For any $x \in L$,

$$\overleftarrow{\mu}\mu(\theta)(x) = \operatorname{Sup}\{\theta(a)\colon (a)^{++} = (x)^{++}, a \in L\} = \theta(x). \tag{27}$$

Thus, θ is a μ-fuzzy filter of L.

Theorem 6. *A fuzzy filter θ of L is a μ-fuzzy filter if and only if*

$$\wedge_{a\in(x)^{++}}\theta(a) \geq \theta(x), \quad \text{for all } x \in L. \tag{28}$$

Proof. Suppose a fuzzy filter θ of L is a μ-fuzzy filter. Then, by Theorem 4, every level subset is a μ-filter of L. Let $x \in L$ such that $\theta(x) = t$. Since θ_t is a μ-filter of L, then $(x)^{++} \subseteq \theta_t$, which implies $a \in \theta_t$ for all $a \in (x)^{++}$. Thus, $\theta(a) \geq \theta(x)$ for each $a \in (x)^{++}$. So,

$$\wedge_{a\in(x)^{++}}\theta(a) \geq \theta(x), \quad \text{for all } x \in L. \tag{29}$$

Suppose conversely that the condition holds. To prove θ is a μ-fuzzy filter, it suffices to show that $\overleftarrow{\mu}\mu(\theta) \subseteq \theta$. For any $x \in L$, $\overleftarrow{\mu}\mu(\theta)(x) = \operatorname{Sup}\{\theta(b)\colon (b)^{++} = (x)^{++}\}$. If $(b)^{++} = (x)^{++}$, then $x \in (b)^{++}$. By the assumption, $\theta(x) \geq \theta(b)$ for each $(b)^{++} = (x)^{++}$. This shows that $\theta(x)$ is an upper bound of $\{\theta(b)\colon (b)^{++} = (x)^{++}\}$. Thus, $\overleftarrow{\mu}\mu(\theta)(x) \leq \theta(x)$ for all $x \in L$. So, $\overleftarrow{\mu}\mu(\theta) \subseteq \theta$. Hence, θ is a μ-fuzzy filter of L.

Let us denote the set of all μ-fuzzy filters of L by $\mathrm{FF}_\mu(L)$.

Theorem 7. *The class $(FF_\mu(L), \wedge, \underline{\vee})$ of all μ-fuzzy filters of L forms a complete distributive lattice with respect to set inclusion.*

Proof. Clearly, $(\mathrm{FF}_\mu(L), \subseteq)$ is a partially ordered set. For $\eta, \theta \in \mathrm{FF}_\mu(L)$, define

$$\begin{aligned}\eta \wedge \theta &= \eta \cap \theta, \\ \eta \underline{\vee} \theta &= \overleftarrow{\mu}\mu(\eta \vee \theta).\end{aligned} \tag{30}$$

Then, clearly $\eta \wedge \theta, \eta \underline{\vee} \theta \in \mathrm{FF}_\mu(L)$. We need to show $\eta \underline{\vee} \theta$ is the least upper bound of $\{\eta, \theta\}$. Since $\theta, \eta \subseteq \eta \vee \theta \subseteq \eta \underline{\vee} \theta$, $\eta \underline{\vee} \theta$ is an upper bound of $\{\mu, \theta\}$. Let λ be any upper bound for η, θ in $\mathrm{FF}_\mu(L)$. Then, $\eta \vee \theta \subseteq \lambda$, which implies that $\overleftarrow{\mu}(\eta \vee \theta) \subseteq \overleftarrow{\mu}\mu(\lambda) = \lambda$. Therefore, $\overleftarrow{\mu}\mu(\eta \vee \theta)$ is the supremum of both $\{\eta, \theta\}$ in $\mathrm{FF}_\mu(L)$. Hence, $(\mathrm{FF}_\mu(L), \wedge, \underline{\vee})$ is a lattice.

We now prove the distributivity. Let $\eta, \theta, \lambda \in \mathrm{FF}_\mu(L)$. Then,

$$\begin{aligned}\eta \underline{\vee} (\theta \cap \lambda) &= \overleftarrow{\mu}\mu(\eta \vee (\theta \cap \lambda)) = \overleftarrow{\mu}\mu((\eta \vee \theta) \cap (\eta \vee \lambda)) \\ &= \overleftarrow{\mu}\mu(\eta \vee \theta) \cap \overleftarrow{\mu}\mu(\eta \vee \lambda) = (\eta \underline{\vee} \theta) \cap (\eta \underline{\vee} \lambda).\end{aligned} \tag{31}$$

Thus, $\mathrm{FF}_\mu(L)$ is a distributive lattice.

Now, we proceed to show the completeness. Since $\{1\}$ and L are μ-filters, $\chi_{\{1\}}$ and χ_L are least and greatest elements of $\mathrm{FF}_\mu(L)$, respectively. Let $\{\theta_i\colon i \in I\} \subseteq \mathrm{FF}_\mu(L)$. Then, $\cap_{i\in I}\theta_i$ is a fuzzy filter of L and $\cap_{i\in I}\theta_i \subseteq \overleftarrow{\mu}\mu(\cap_{i\in I}\theta_i)$.

$$\begin{aligned}\bigcap_{i\in I} \theta_i \subseteq \theta_i, \quad \forall i \in I &\Longrightarrow \overleftarrow{\mu}\mu\Big(\bigcap_{i\in I}\theta_i\Big) \subseteq \theta_i, \\ \forall i \in I &\Longrightarrow \overleftarrow{\mu}\mu\Big(\bigcap_{i\in I}\theta_i\Big) \subseteq \bigcap_{i\in I}\theta_i.\end{aligned} \tag{32}$$

Thus, $\overleftarrow{\mu}\mu(\cap_{i\in I}\theta_i) = \cap_{i\in I}\theta_i$. So, $\mathrm{FF}_\mu(L), \wedge, \vee$ is a complete distributive lattice.

Theorem 8. *The set $FF_\mu(L)$ is isomorphic to the lattice of fuzzy ideals of $A^+(L)$.*

Proof. Define

$$\begin{aligned}f\colon \mathrm{FF}_\mu(L) &\longrightarrow \mathrm{FI}(A^+(L)), \\ f(\eta) &= \mu(\eta), \quad \forall \theta \in \mathrm{FF}_\mu(L).\end{aligned} \tag{33}$$

Let $\eta, \theta \in \mathrm{FF}_\mu(L)$ and $f(\eta) = f(\theta)$. Then, $\mu(\eta) = \mu(\theta)$. Thus, $\overleftarrow{\mu}\mu(\eta) = \overleftarrow{\mu}\mu(\theta)$. So, $\eta = \theta$. Hence, f is one to one.

Let $\lambda \in \mathrm{FI}(A^+(L))$. Then, by Lemma 3, $\overleftarrow{\mu}(\lambda)$ is a fuzzy filter of L. Now, we proceed to show that $\overleftarrow{\mu}(\lambda)$ is a μ-fuzzy filter of L. Let $x \in L$. Then, $\overleftarrow{\mu}\mu(\overleftarrow{\mu}(\lambda))(x) = \mu\overleftarrow{\mu}(\lambda)((x)^{++})$. Thus, by Lemma 6, we get that $\mu\overleftarrow{\mu}(\lambda)((x)^{++}) = \overleftarrow{\mu}(\eta)(x)$. So, $\overleftarrow{\mu}(\lambda) = \overleftarrow{\mu}\mu(\overleftarrow{\mu}(\lambda))$. Thus, for each $\lambda \in \mathrm{FI}(A^+(L))$, $f(\overleftarrow{\mu}(\lambda)) = \lambda$. Therefore, f is onto.

Now, for any $\eta, \theta \in \mathrm{FI}_\mu(L)$, $f(\eta \underline{\vee} \theta) = f(\overleftarrow{\mu}\mu t(\eta \vee \theta)) = \mu(\overleftarrow{\mu}\mu t(\eta \vee \theta)) = \mu(\eta \vee \theta) = \mu(\eta) \vee \mu(\theta) = f(\eta) \vee f(\theta)$. Similarly, $f(\eta \cap \theta) = f(\eta) \cap f(\theta)$. Therefore, f is an isomorphism of $\mathrm{FF}_\mu(L)$ onto the lattice of fuzzy filters of $A^+(L)$.

Theorem 9. *The following are equivalent for each nonconstant μ-fuzzy filter λ of L.*

(1) For all $\theta, \eta \in \mathrm{FF}(L)$,

$$\theta \cap \eta \subseteq \lambda \Rightarrow \theta \subseteq \lambda \text{ or } \eta \subseteq \lambda. \tag{34}$$

(2) For any fuzzy points x_γ and y_β of L,

$$x_\gamma + y_\beta \subseteq \lambda \Rightarrow x_\gamma \subseteq \lambda \text{ or } y_\beta \subseteq \lambda. \quad (35)$$

(3) For all $\theta, \eta \in FF_\mu(L)$,

$$\theta \cap \eta \subseteq \lambda \Rightarrow \theta \subseteq \lambda \text{ or } \eta \subseteq \lambda. \quad (36)$$

Proof

1⟹2. Let $x, y \in L$ such that $x_\gamma + y_\beta \subseteq \lambda$. Then, $[x_\gamma + y_\beta) \subseteq \lambda$. Since L is a distributive lattice, by dual of Lemma 2, we have $[x_\gamma) \cap [y_\beta) \subseteq \lambda$. Since $[x_\gamma)$ and $[y_\beta)$ are fuzzy filters of L, by the assumption, $[x_\gamma) \subseteq \lambda$ or $[y_\beta) \subseteq \lambda$. This shows that $x_\gamma \subseteq \lambda$ or $y_\beta \subseteq \lambda$.

2⟹3. Let $\theta, \eta \in FF_\mu(L)$ such that $\theta \cap \eta \subseteq \lambda$. Now, we need to show $\theta \subseteq \lambda$ or $\eta \subseteq \lambda$. Suppose not. Then, $\theta \nsubseteq \lambda$ and $\eta \nsubseteq \lambda$, which implies there exist $x, y \in L$ such that $\theta(x) > \lambda(x)$ and $\eta(y) > \lambda(y)$. Put $\theta(x) = \gamma$ and $\eta(y) = \beta$. Then, $x_\gamma \nsubseteq \lambda$ and $y_\beta \nsubseteq \lambda$. Since $x_\gamma + y_\beta \subseteq \theta \cap \eta$, we have $x_\gamma + y_\beta \subseteq \lambda$. By the assumption, we get that $x_\gamma \subseteq \lambda$ or $y_\beta \subseteq \lambda$, which is a contradiction. Thus, $\theta \subseteq \lambda$ or $\eta \subseteq \lambda$.

3⟹1. Suppose $\theta, \eta \in FF(L)$ such that $\theta \cap \eta \subseteq \lambda$. Then, by Corollary 1 we have $\overleftarrow{\mu}\mu(\theta) \cap \overleftarrow{\mu}\mu(\eta) \subseteq \lambda$. Since $\overleftarrow{\mu}\mu(\theta)$ and $\overleftarrow{\mu}\mu(\eta)$ are μ-fuzzy filters, by the assumption, we get that $\overleftarrow{\mu}\mu(\theta) \subseteq \lambda$ or $\overleftarrow{\mu}\mu(\eta) \subseteq \lambda$, which implies $\theta \subseteq \lambda$ or $\eta \subseteq \lambda$.

Definition 7. By a prime μ-fuzzy filter, we mean a non-constant μ-fuzzy filter of L satisfying (1) and hence all of the conditions of Theorem 9.

We have proved in Theorem 8 that there is an order isomorphism between the class of μ-fuzzy filters and the set of fuzzy ideals of $A^+(L)$. Now, we show that there is an isomorphism between the prime μ-fuzzy filters and the prime fuzzy ideals of the lattice of coannihilators.

Theorem 10. *There is an isomorphism between the prime μ-fuzzy filters and the prime fuzzy ideals of the lattice of coannihilator.*

Proof. By Theorem 8, the map f is an isomorphism from $FF_\mu(L)$ into $FI(A^+(L))$. Let σ be a prime μ-fuzzy filter of L. Then, $\mu(\sigma) \in FI(A^+(L))$. Now, we prove $\mu(\sigma)$ is a prime fuzzy ideal of $FI(A^+(L))$. Let $\theta, \eta \in FI(A^+(L))$ such that $\theta \cap \eta \subseteq \mu(\sigma)$. Since f is onto, there exist $\lambda, \gamma \in FF_\mu(L)$ such that $f(\lambda) = \theta$ and $f(\gamma) = \eta$. Thus, $\mu(\lambda \cap \gamma) \subseteq \mu(\sigma)$. Since $\overleftarrow{\mu}$ is an isotone, we have $\overleftarrow{\mu}\mu(\lambda \cap \gamma) \subseteq \overleftarrow{\mu}\mu(\sigma)$. Thus, $\lambda \cap \gamma \subseteq \sigma$. Since σ is a prime fuzzy filter, either $\lambda \subseteq \sigma$ or $\gamma \subseteq \sigma$. This shows that either $\mu(\lambda) \subseteq \mu(\sigma)$ or $\mu(\gamma) \subseteq \mu(\sigma)$. Thus, $\theta \subseteq \mu(\sigma)$ or $\eta \subseteq \mu(\sigma)$. Hence, $\mu(\sigma)$ is a prime fuzzy ideal of $A^+(L)$.

Conversely, suppose that θ is a prime fuzzy ideal in $A^+(L)$. Since f is onto, there exists a μ-fuzzy filter σ in $FF_\mu(L)$ such that $\theta = \mu(\sigma)$. Let $\eta, \lambda \in FF(L)$ such that $\eta \cap \lambda \subseteq \sigma$. Since μ is an isotone, we get $\mu(\eta \cap \lambda) \subseteq \mu(\sigma) = \theta$. Thus, $\mu(\eta) \cap \mu(\lambda) \subseteq \mu(\sigma)$. Since $\mu(\sigma)$ is a prime fuzzy ideal of $A^+(L)$, either $\mu(\eta) \subseteq \mu(\sigma)$ or $\mu(\lambda) \subseteq \mu(\sigma)$. This implies $\eta \subseteq \overleftarrow{\mu}\mu(\sigma)$ or $\lambda \subseteq \overleftarrow{\mu}\mu(\sigma)$. Since σ is a μ-fuzzy filter, we get $\eta \subseteq \sigma$ or $\lambda \subseteq \sigma$. Thus, σ is prime fuzzy filter in $FF_\mu(L)$. So, the prime μ-fuzzy filters correspond to prime fuzzy ideals of $A^+(L)$.

Theorem 11. *Let θ be a μ-fuzzy filter of L and η be a fuzzy ideal of L such that $\theta \cap \eta \leq \alpha$, $\alpha \in [0, 1)$. Then, there exists a prime μ-fuzzy filter λ of L such that $\theta \subseteq \lambda$ and $\lambda \cap \eta \leq \alpha$.*

Proof. Put $P = \{\sigma \in FF_\mu(L)\colon \sigma \subseteq \eta \text{ and } \eta \cap \sigma \leq \alpha\}$. Since $\theta \in \mathscr{P}$, $\mathscr{P}$ is nonempty, and it forms a poset together with the inclusion ordering of fuzzy sets. Let $\mathscr{A} = \{\theta_i\}_{i \in I}$ be any chain in $\mathscr{P}$. Then, clearly $\cup_{i \in I}\theta_i$ is a μ-fuzzy filter. Since $\theta_i \cap \eta \leq \alpha$ for each $i \in I$, we get that $(\cup_{i \in I}\theta_i) \cap \eta \leq \alpha$. Thus, $\cup_{i \in I}\theta_i \in \mathscr{A}$. By applying Zorn's lemma, we get a maximal element, say $\sigma \in \mathscr{P}$; that is, σ is a μ-fuzzy filter of L such that $\theta \subseteq \sigma$ and $\sigma \cap \eta \leq \alpha$.

Now, we proceed to show σ is a prime fuzzy filter. Assume that σ is not prime fuzzy filter. Let $\gamma_1 \cap \gamma_2 \subseteq \sigma$ such that $\gamma_1 \nsubseteq \sigma$ and $\gamma_2 \nsubseteq \sigma$, $\gamma_1, \gamma_2 \in FF(L)$. If we put $\sigma_1 = \overleftarrow{\mu}\mu(\gamma_1 \vee \sigma)$ and $\sigma_2 = \overleftarrow{\mu}\mu(\gamma_2 \vee \sigma)$, then both σ_1 and σ_2 are μ-fuzzy filters of L properly containing σ. Since σ is maximal in $\mathscr{P}$, we get $\sigma_1, \sigma_2 \notin \mathscr{P}$. Thus, $\sigma_1 \cap \eta \nleq \alpha$ and $\sigma_2 \cap \eta \nleq \alpha$. This implies there exist $x, y \in L$ such that $(\sigma_1 \cap \eta)(x) > \alpha$ and $(\sigma_2 \cap \eta)(y) > \alpha$, which implies $((\sigma_1 \cap \sigma_2) \cap \eta)(x \wedge y) > \alpha \Rightarrow (\overleftarrow{\mu})\mu(\sigma \vee (\gamma_1 \wedge \gamma_2))$ $(x \wedge y) \wedge \eta(x \wedge y) > \alpha$. This shows that $(\theta \cap \eta)(x \wedge y) > \alpha$. This is a contradiction. Thus, σ is prime μ-fuzzy filter of L.

Corollary 3. *Let θ be a μ-fuzzy filter of L, $a \in L$ and $\alpha \in [0, 1)$. If $\theta(a) \leq \alpha$, then there exists a prime μ-fuzzy filter η of L such that $\theta \subseteq \eta$ and $\eta(a) \leq \alpha$.*

Proof. Put $\mathscr{P} = \{\sigma \in FF_\mu(L)\colon \sigma \subseteq \eta \text{ and } \eta \cap \sigma \leq \alpha\}$. Since $\theta \in \mathscr{P}$, $\mathscr{P}$ is nonempty, and it forms a poset together with the inclusion ordering of fuzzy sets. Let $\mathscr{A} = \{\theta_i\}_{i \in I}$ be any chain in $\mathscr{P}$. Clearly, $\cup_{i \in I}\theta_i$ is a μ-fuzzy filter. Since $\theta_i(a) \leq \alpha$ for each $i \in I$, α is an upper bound of $\{\theta_i(a)\colon i \in I\}$. Thus, $\cup_{i \in I}\theta_i(a) \leq \alpha$. So, $\cup_{i \in I}\theta_i$ is a μ-fuzzy filter containing θ and $\cup_{i \in I}\theta_i(a) \leq \alpha$. Hence, $\cup_{i \in I}\theta_i \in \mathscr{P}$. By applying Zorn's lemma, we get a maximal element, say $\sigma \in \mathscr{P}$; that is, σ is a μ-fuzzy filter of L such that $\theta \subseteq \sigma$ and $\sigma(a) \leq \alpha$.

Now, we proceed to show σ is a prime fuzzy filter. Assume that σ is not prime fuzzy filter. Let $\gamma_1 \cap \gamma_2 \subseteq \sigma$ and $\gamma_1 \nsubseteq \sigma$ and $\gamma_2 \nsubseteq \sigma$, $\gamma_1, \gamma_2 \in FF(L)$. If we put $\sigma_1 = \overleftarrow{\mu}\mu(\gamma_1 \vee \sigma)$ and $\sigma_2 = \overleftarrow{\mu}\mu(\gamma_2 \vee \sigma)$, then both σ_1 and σ_2 are μ-fuzzy filters of L properly containing σ. Since σ is maximal in $\mathscr{P}$, we get $\sigma_1, \sigma_2 \notin \mathscr{P}$. Thus, $\sigma_1(a) > \alpha$ and $\sigma_2(a) > \alpha$. Now, (sigma$_1 \cap \sigma_2)(a) = \overleftarrow{\mu}\mu((\gamma_1 \vee \sigma) \cap (\gamma_2 \vee \sigma))(a) = \overleftarrow{\mu}\mu((\gamma_1 \cap \gamma_2) \ \vee \sigma))(a) = \sigma(a) > \alpha$. This is a contradiction. Hence, σ is prime μ-fuzzy filter.

Corollary 4. *Any μ-fuzzy filter of L is the intersection of all prime μ-fuzzy filters containing it.*

Proof. Let θ be a proper μ-fuzzy filter of L. Consider the following.

$$\lambda = \cap\{\eta\colon \eta \text{ is a prime } \mu - \text{fuzzy filter and } \theta \subseteq \eta\}. \quad (37)$$

Clearly, $\theta \subseteq \lambda$. Assume that $\lambda \nsubseteq \theta$. Then, there is $a \in L$ such that $\lambda(a) > \theta(a)$. Let $\theta(a) = \alpha$. Consider the set

$$\mathscr{P} = \left\{\eta \in \mathrm{FI}_\mu(L)\colon \theta \subseteq \eta \text{ and } \eta(a) \leq \alpha\right\}. \quad (38)$$

By the above corollary, we can find a prime μ-fuzzy filter γ of L such that $\theta \subseteq \gamma$ and $\gamma(a) \leq \alpha$. This implies $\lambda \subseteq \gamma$. This shows that $\lambda \leq \alpha$, which is a contradiction. Thus, $\lambda \subseteq \theta$. So, $\lambda = \theta$.

4. The Space of Prime μ-Fuzzy Filters

In this section, we study the space of prime μ-fuzzy filters of a distributive lattice and some properties of the space also.

Let X_μ be the set of all prime μ-fuzzy filters of a distributive lattice. Let $V(\theta) = \left\{\eta \in X_\mu\colon \theta \subseteq \eta\right\}$, where θ is a fuzzy subset of L and $X(\theta) = \left\{\eta \in X_\mu\colon \theta \nsubseteq \eta\right\} = X_\mu - V(\theta)$. We let $\mu_* = \mu_1$, i.e., $\mu_* = \{x \in L\colon \mu(x) = 1\}$.

Lemma 8. *For any fuzzy filters λ and θ of L, we have*

(1) $\lambda \subseteq \theta \Rightarrow X(\lambda) \subseteq X(\theta)$

(2) $X(\lambda \vee \theta) = X(\lambda) \cup X(\theta)$

(3) $X(\lambda \cap \theta) = X(\lambda) \cap X(\theta)$

Proof

(1) Let $\lambda \subseteq \theta$ and $\eta \in X(\lambda)$. Then, $\lambda \nsubseteq \eta$ and $\theta \nsubseteq \eta$. Thus, $\eta \in X(\theta)$.

(2) Since $\lambda, \theta \subseteq \lambda \vee \theta$, by (1), we have $X(\lambda) \cup X(\theta) \subseteq X(\lambda \vee \theta)$. Now, we proceed to show the other inclusion; let $\eta \in X(\lambda \vee \theta)$. Then, $\lambda \vee \theta \nsubseteq \eta$. This shows that either $\lambda \nsubseteq \eta$ or $\theta \nsubseteq \eta$. So, $\eta \in X(\lambda) \cup X(\theta)$. Hence, $X(\lambda \vee \theta) = X(\lambda) \cup X(\theta)$.

(3) By (1), we have $X(\lambda \cap \theta) \subseteq X(\lambda) \cap X(\theta)$. On the other hand, let $\eta \in X(\lambda) \cap X(\theta)$. Then, $\lambda \nsubseteq \eta$ and $\theta \nsubseteq \eta$. Since η is a prime fuzzy filter, we get that $\lambda \cap \theta \nsubseteq \eta$. This shows that $\eta \in X(\lambda \cap \theta)$. Thus, $X(\lambda \cap \theta) = X(\lambda) \cap X(\theta)$.

Lemma 9. *Let θ be a fuzzy subset of L. Then, $X(\theta) = X([\theta])$.*

Proof. To prove our claim, it suffices to show $X([\theta]) \subseteq X(\theta)$. Let $\eta \in X([\theta])$. Then, $[\theta] \nsubseteq \eta$. We need to show $\theta \nsubseteq \eta$. Suppose not. Then, $\theta \subseteq \eta$, which implies that $[\theta] \subseteq \eta$, which is a contradiction. Thus, $\theta \nsubseteq \eta$. So, $X(\theta) = X([\theta])$.

Theorem 12. *Let $x, y \in L$ and $\beta \in (0,1]$. Then,*

(1) $X((x \wedge y)_\beta) = X(x_\beta) \cup X(y_\beta)$

(2) $X((x \vee y)_\beta) = X(x_\beta) \cap X(y_\beta)$

(3) $\cup_{x \in L, \beta \in (0,1]} X(x_\beta) = X_\mu$

Proof

(1) If $\lambda \in X(x_\beta) \cup X(y_\beta)$, then either $x_\beta \nsubseteq \lambda$ or $y_\beta \nsubseteq \lambda$. This shows that $\beta > \lambda(x)$ or $\beta > \lambda(y)$. Thus, $\beta > \lambda(x) \wedge \lambda(y) = \lambda(x \wedge y)$. So, $(x \wedge y)_\beta \nsubseteq \lambda$. Hence, $\lambda \in X((x \wedge y)_\beta)$.

On the other hand, let $\lambda \in X(x \wedge y)_\beta$. Then, $\beta > \lambda(x \wedge y) = \lambda(x) \wedge \lambda(y)$. This implies either $x_\beta \nsubseteq \lambda$ or $y_\beta \nsubseteq \lambda$. Thus, $\lambda \in X(x_\beta) \cup X(y_\beta)$.

(2) If $\lambda \in X(x_\beta) \cap X(y_\beta)$, then $x_\beta \nsubseteq \lambda$ and $y_\beta \nsubseteq \lambda$. This implies $\beta > \lambda(x)$ and $\beta > \lambda(y)$. This shows that $x, y \notin \lambda_*$. Since λ is prime fuzzy filter, card Im $\lambda = 2$ and λ_* is prime. Thus, $x \vee y \notin \lambda_*$, which implies $\beta > \lambda(x \vee y)$. Thus, $(x \vee y)_\beta \nsubseteq \lambda$ and hence $X(x_\beta) \cap X(y_\beta) \subseteq X((x \wedge y)_\beta)$.

Conversely, let $\lambda \in X((x \vee y)_\beta)$. Then, $(x \vee y)_\beta \nsubseteq \lambda$, which implies $\beta > \lambda(x \vee y) \geq \lambda(x) \vee \lambda(y)$. Thus, $\beta > \lambda(x)$ and $\beta > \lambda(y)$. This shows that $x_\beta \nsubseteq \lambda$ and $y_\beta \nsubseteq \lambda$. Thus, $\lambda \in X(x_\beta) \cap X(y_\beta)$. So, $X((x \vee y)_\beta) \subseteq X(x_\beta) \cap X(y_\beta)$. Therefore, $X((x \vee y)_\beta) = X(x_\beta) \cap X(y_\beta)$.

(3) Clearly, $\cup_{x \in L, \beta \in (0,1]} X(x_\beta) \subseteq X_\mu$. Let $\lambda \in X_\mu$. Then, $\mathrm{Im}\,\lambda = \{1, \gamma\}$, $\gamma \in [0,1)$. This implies there is $x \in L$ such that $\lambda(x) = \gamma$. If we take some $\beta \in (0,1)$ such that $\beta > \gamma$, then $x_\beta \nsubseteq \lambda$. Thus, $X_\mu \subseteq \cup_{x \in L, \beta \in (0,1]} X(x_\beta)$. So, $X_\mu = \cup_{x \in L, \beta \in (0,1]} X(x_\beta)$.

Lemma 10. *Let $\beta_1, \beta_1 \in (0,1]$; $\beta = min\{\beta_1, \beta_2\}$; and $x, y \in L$. Then,*

$$X\left(x_{\beta_1}\right) \cap X\left(y_{\beta_2}\right) = X\left((x \vee y)_\beta\right). \quad (39)$$

Proof. Let $\lambda \in X(x_{\beta_1}) \cap X(y_{\beta_2})$. Then, $x_{\beta_1} \nsubseteq \lambda$ and $x_{\beta_2} \nsubseteq \lambda$. This implies that $\beta_1 > \lambda(x)$ and $\beta_2 > \lambda(y)$. Since λ_* is prime filter and $x, y \notin \lambda_*$, we have $x \vee y \notin \lambda_*$ and $\lambda(x) = \lambda(y) = \lambda(x \vee y)$. This shows that $\beta > \lambda(x \vee y)$. Thus, $(x \vee y)_\beta \nsubseteq \lambda$. So, $\lambda \in X((x \vee y)_\beta)$. To show the other inclusion, let $\lambda \in X((x \vee y)_\beta)$. Then, $\beta > \lambda(x \vee y) \geq \lambda(x) \vee \lambda(y)$. This implies $\beta_1 > \lambda(x)$ and $\beta_2 > \lambda(y)$. Thus, $x_{\beta_1} \nsubseteq \lambda$ and $y_{\beta_2} \nsubseteq \lambda$. So, $\lambda \in X(x_{\beta_1}) \cap X(y_{\beta_2})$. Hence, $X(x_{\beta_1}) \cap X(y_{\beta_2}) = X((x \vee y)_\beta)$.

Lemma 11. *Let $\{\theta_i\colon i \in I\}$ be any family of fuzzy filters of L. Then,*

$$\bigcap_{i \in I} V(\theta_i) = V\left(\left[\bigcup_{i \in I} \theta_i\right]\right). \quad (40)$$

Proof. Since $\theta_i \subseteq [\cup_{i \in I} \theta_i)$ for each $i \in I$, we have $V([\cup_{i \in I} \theta_i)) \subseteq V(\theta_i)$ for each $i \in I$. Thus, $V([\cup_{i \in I} \theta_i)) \subseteq \cup_{i \in I} V(\theta_i)$.

Conversely, let $\lambda \in \cup_{i \in I} V(\theta_i)$. Then, $\lambda \in V(\theta_i)$ for each $i \in I$. This implies $\theta_i \subseteq \mu$. Thus, for any $x \in L$, $\mu(x)$ is an upper bound of $\{\theta_i(x)\colon i \in I\}$. This implies that $\mathrm{Sup}\{\theta_i(x)\colon i \in I\} \leq \lambda(x)$. This shows that $\cup_{i \in I} \theta_i \subseteq \lambda$ and $[\cup_{i \in I} \theta_i) \subseteq \lambda$. So, $\mu \in V([\cup_{i \in I} \theta_i))$. Thus, $\cap_{i \in I} V(\theta_i) \subseteq V((\cup_{i \in I} \theta_i])$. Hence, $\cap_{i \in I} V(\theta_i) = V([\cup_{i \in I} \theta_i))$.

Theorem 13. *The collection $\mathcal{T} = \{X(\theta)\colon \theta$ is a fuzzy filter of $L\}$ is a topology on X_μ.*

Proof. First, we define two fuzzy subsets η_1, η_2 of L as follows: $\eta_1(x) = 0$ and $\eta_2(x) = 1$ for all $x \in L$. Then, $[\eta_1)$ and η_2 are fuzzy filters of L. Since $[\eta_1) \subseteq \lambda$, for all $\lambda \in X_\mu$, we get that $V([\eta_1)) = X_\mu$. This shows that $X(\eta_1) = \phi$. Since each $\lambda \in X_\mu$ is nonconstant, $\eta_2 \nsubseteq \lambda$ for all $\lambda \in X_\mu$. So $X(\eta_2) = X_\mu$. Hence, ϕ, $X_\mu \in \mathcal{T}$.

Next, let $X(\theta_1), X(\theta_2) \in \mathcal{T}$. Then, by Lemma 8, we get that $X(\theta_1) \cap X(\theta_2) = X(\theta_1 \cap \theta_2)$. This shows that $\mathcal{T}$ is closed under finite intersection.

Now, we proceed to show that $\mathcal{T}$ is closed under arbitrary union. Let $\{\theta_i\colon i \in I\}$ be any family of fuzzy filters of L. Then, by Lemma 11 we have

$$\bigcap_{i\in I} V(\theta_i) = V\left(\left[\bigcup_{i\in I} \theta_i\right)\right), \quad (41)$$

which implies $\cup_{i\in I} X(\theta_i) = X([\cup_{i\in I}\theta_i))$. Thus, by Lemma 9, we get that

$$X\left(\bigcup_{i\in I} \theta_i\right) = X\left(\left[\bigcup_{i\in I} \theta_i\right)\right). \quad (42)$$

So, $\mathcal{T}$ is closed under arbitrary union. Therefore, $\mathcal{T}$ is a topology on X_μ. The space $(X_\mu, \mathcal{T})$ will be called the space of prime μ-fuzzy filters in L.

In the above theorem, we proved that the family of $X(\theta)$ is a topology on X_μ. In the following result, we show that the set of all open sets of the form $X(x_\beta)$ is a basis for the topology on X_μ.

Theorem 14. *The collection $\mathcal{B} = \{X(x_\beta)\colon x \in L, \beta \in (0,1]\}$ forms base for some topology X_μ.*

Proof. Let $X(\theta)$ be any open set in X_μ and $\lambda \in X(\theta)$. Then, $\theta \nsubseteq \lambda$ and there is $x \in L$ such that $\theta(x) > \lambda(x)$. Put $\theta(x) = \beta$; then $x_\beta \subseteq \theta$ and $\lambda \in X(x_\beta)$. To show $X(x_\beta) \subseteq X(\theta)$, let $\eta \in X(x_\beta)$. Then, $x_\beta \nsubseteq \eta$ and $\theta(x) > \eta(x)$. This shows that $\eta \in X(\theta)$. Thus, $\lambda \in X(x_\beta) \subseteq X(\theta)$. Hence, for any open set $X(\theta)$ in X_μ we can find $X(x_\beta)$ in $\mathcal{B}$ such that $X(x_\beta) \subseteq X(\theta)$. Therefore, $\mathcal{B}$ is a base for $\mathcal{T}$.

Theorem 15. *The space X_μ is a T_0-space.*

Proof. Take any two different elements η and θ in X_μ. Then, either $\eta \nsubseteq \theta$ or $\theta \nsubseteq \eta$. Without loss of generality, we can assume that $\eta \nsubseteq \theta$. Then, $\theta \in X(\eta)$ and $\eta \notin X(\eta)$. Thus, X_μ is a T_0-space.

Theorem 16. *For any fuzzy filter η of L, $X(\eta) = X(\overleftarrow{\mu}\mu(\eta))$.*

Proof. For any fuzzy filter η of L, we have $\eta \subseteq \overleftarrow{\mu}\mu(\eta)$ and $X(\eta) \subseteq X(\overleftarrow{\mu}\mu(\eta))$. Now we proceed to show the other inclusion; let $\theta \in X(\overleftarrow{\mu}\mu(\eta))$. Then, $\overleftarrow{\mu}\mu(\eta)) \nsubseteq \theta$. Suppose $\theta \notin X(\eta)$; then $\eta \subseteq \theta$. This implies $\overleftarrow{\mu}\mu(\eta) \subseteq \overleftarrow{\mu}\mu(\theta)) = \theta$, which is impossible. Thus, $\theta \in X(\eta)$ and hence $X(\eta) = X(\overleftarrow{\mu}\mu(\eta))$.

In the following result, we show that there is a one-to-one correspondence between the class of μ-fuzzy filters and the lattice of all open sets in X_μ.

Theorem 17. *The lattice $FF_\mu(L)$ is isomorphic with the lattice of all open sets in X_μ.*

Proof. The lattice of all open sets in X_μ is $(\mathcal{T}, \cap, \cup)$. Define the mapping

$$f\colon \mathrm{FF}_\mu(L) \longrightarrow \mathcal{T} \text{ by } f(\lambda) = X(\lambda), \quad \text{for all } \lambda \in X_\mu. \quad (43)$$

Since $X(\lambda) = X(\overleftarrow{\mu}\mu t(\lambda))$ and $\overleftarrow{\mu}\mu(\lambda)$ is a μ-fuzzy filter, every open subset of X_μ is of the form $X(\theta)$ for some $\theta \in \mathrm{FF}_\mu(L)$. This shows that the map is onto.

Let $f(\lambda) = f(\theta)$. Now, we need to show $\lambda = \theta$. Suppose not. Then, $\lambda \neq \theta$, which implies that there is $x \in L$ such that either $\lambda(x) < \theta(x)$ or $\theta(x) < \lambda(x)$. Without loss of generality, we can assume that $\lambda(x) < \theta(x)$. Put $\lambda(x) = \beta$. Then by Corollary 3, we can find a prime μ-fuzzy filter η such that $\lambda \subseteq \eta$ and $\eta(x) \leq \beta$. Thus, $\eta \notin X(\lambda)$ and $\theta \nsubseteq \eta$. So, $\eta \notin X(\lambda)$ and $\eta \in X(\theta)$. This is a contradiction. Hence, $\lambda = \theta$.

Now, we show that f is homomorphism. Let $\lambda, \theta \in \mathrm{FF}_\mu(L)$. Then,

$$f(\lambda \underline{\vee} \theta) = X(\overleftarrow{\mu}\mu(\lambda \vee \theta)) = X(\lambda \vee \theta) = f(\lambda) \cup f(\theta). \quad (44)$$

Similarly, $f(\lambda \cap \theta) = f(\lambda) \cap f(\theta)$. This shows that f is a homomorphism. Hence, f is an isomorphism.

For any fuzzy subset η of L, $X(\theta) = \{\eta \in X_\mu\colon \theta \nsubseteq \eta\}$ is an open set of X_μ and $V(\theta) = \{\eta \in X_\mu\colon \theta \subseteq \eta\} = X_\mu - V(\theta)$ is a closed set of X_μ. In the following result, we prove the closure of a fuzzy set.

Theorem 18. *For any family $\mathcal{F} \subseteq X_\mu$, closure of $\mathcal{F}$ is given by $\overline{\mathcal{F}} = V(\cap_{\lambda\in\mathcal{F}}\lambda)$.*

Proof. We know that closure of $\mathcal{F}$ is the smallest closed set containing $\mathcal{F}$. To prove our claim, it is enough to show that $V(\cap_{\lambda\in\mathcal{F}}\lambda)$ is the smallest closed set containing $\mathcal{F}$. Since the set of all μ-fuzzy filter is a complete distributive lattice, $\cap_{\lambda\in\mathcal{F}}\lambda$ is a μ-fuzzy filter and $V(\cap_{\lambda\in\mathcal{F}}\lambda)$ is a closed set in X_μ. If $\eta \in \mathcal{F}$; then $\cap_{\mu\in\mathcal{F}}\lambda \subseteq \eta$. Thus, $\eta \in V(\cap_{\lambda\in\mathcal{F}}\lambda)$. This implies that $\mathcal{F} \subseteq V(\cap_{\lambda\in\mathcal{F}}\lambda)$. Let $V(\theta)$ be any closed set in X_μ containing $\mathcal{F}$. Then, $\theta \subseteq \lambda$, for each $\lambda \in \mathcal{F}$. Thus, $\theta \subseteq \cap_{\lambda\in\mathcal{F}}\lambda$ and $V(\cap_{\lambda\in\mathcal{F}}\lambda) \subseteq V(\theta)$. So, $V(\cap_{\lambda\in\mathcal{F}}\lambda)$ is the smallest closed set containing $\mathcal{F}$. Hence, $\overline{\mathcal{F}} = V(\cap_{\lambda\in\mathcal{F}}\lambda)$.

5. Conclusion

In this work, we studied the concept of μ-fuzzy filters of a distributive lattice. We proved that the set of all μ-fuzzy filters of a distributive lattice forms a complete distributive lattice isomorphic to the set of all fuzzy ideals of $A^+(L)$. We observed that every μ-fuzzy filter is the intersection of all μ-fuzzy filters containing it. We also studied the space of all prime μ-fuzzy filters in a distributive lattice. Our future work will focus on α-fuzzy ideals of a C-algebra.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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