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Research Article

Fuzzy Ideal Supra Topological Spaces

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In this paper, we introduce the notion of fuzzy ideals in fuzzy supra topological spaces. The concept of a fuzzy s-local function is also introduced here by utilizing the s-neighbourhood structure for a fuzzy supra topological space. These concepts are discussed with a view to find new fuzzy supra topologies from the original one. The basic structure, especially a basis for such generated fuzzy supra topologies, and several relations between different fuzzy ideals and fuzzy supra topologies are also studied here. Moreover, we introduce a fuzzy set operator Ψ_S and study its properties. Finally, we introduce some sets of fuzzy ideal supra topological spaces (fuzzy $*$-supra dense-in-itself sets, fuzzy S^*-supra closedsets, fuzzy $*$-supra perfect sets, fuzzy regular-I-supra closedsets, fuzzy-I-supra opensets, fuzzy semi-I-supra opensets, fuzzy pre-I-supra opensets, fuzzy α-I-supra opensets, and fuzzy β-I-supra opensets) and study some characteristics of these sets, and then, we introduce some fuzzy ideal supra continuous functions.

1. Introduction

The concept of fuzzy sets is an important concept in many fields. The concept was introduced by in 1965 [1]. The idea was welcomed because it addresses the uncertainty, something classical Cantor set theory could not address. Despite of some criticism expressed in the beginning by some specialists of mathematical logic, it has become an important subject in various fields and sciences. Zadeh writes in [1], "The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and potentially, may prove to have a much wider scope of applicability particularly in the fields of pattern classification and information processing."

Fuzzy set theory provides a natural way to deal with inaccuracy and a strict mathematical framework for the study of uncertain phenomena and concepts. It can also be considered as a modeling language, well suited for situations in which fuzzy relations criteria and phenomena exist. Despite the slow growth and progress of fuzzy set theory before the mid-1970s, the theory developed greatly afterward. This was caused by the first successful application of the theory to technological processes, in particular to systems based on ambiguous control rules called fuzzy control and boosted the interest in this area considerably.

The concept of general topology is one of the most important mathematical topics and has wide applications in many applied sciences and mathematical subjects. The notion of fuzzy sets naturally plays a very significant role in the study of fuzzy topology introduced by Chang in 1968 [2]. Pu and Liu in 1980 [3] introduced the concept of quasi-coincidence and q-neighbourhoods by which the extensions of functions in fuzzy setting can very interestingly and effectively be carried out. The concept of an ideal in a topological space was first introduced by Kuratowski in 1966 [4] and Vaidyanathswamy in 1945 [5]. They also defined local functions in an ideal topological space. Furthermore, Jankovic and Hamlet in 1990 [6] studied the properties of ideal topological spaces and introduced another operator called the Ψ operator. They have also obtained a new topology from the original ideal topological space. Using the local function, they defined a Kuratowski closure operator in the new topological space.

The concept of supra topology was introduced by Mashhour et al. in 1983 [7]. It is fundamental with respect to the investigation of general topological spaces. In 2016, Al-shami [8] discussed the concepts of compactness and separation axioms on supra topological spaces. Then, Al-Shami

[9, 10] and Al-shami et al. [11] presented new types of supra compact spaces using supra α-open, supra semiopen, and supra preopensets. Later, the authors of [12–15] employed some generalizations of supra opensets to investigate several kinds of supra limit points of a set and supra T_i spaces $(i = 0, 1, 2, 3, 4)$. In fact, they provided many interesting examples to show the validity of the obtained results. Recently, Al-shami [16] have defined supra paracompact spaces, and Assad et al. [17] have studied γ operation on supra topological spaces.

Abd El-Monsef and Ramadan in 1987 [18] introduced the concept of fuzzy supra topological as a natural generalization of the notion of supra topology spaces. In addition to that, some properties of the concept of ideal supra topological spaces are obtained by Modak and Mistry in 2012 [19]. In 2015 [20], further properties of ideal supra topological spaces are investigated.

In this paper, we introduce the notion of fuzzy ideals in fuzzy supra topological spaces.

Section-wise description of the work carried out in this paper is given. Beginning with an introduction, necessary notation and preliminaries have been given.

In Section 3, the concept of a fuzzy s-local function is also introduced here by utilizing.

In Section 4, we give the s-neighbourhood structure for a fuzzy supra topological space. These concepts are discussed with a view to find new fuzzy supra topologies from the original one. The basic structure, especially a basis for such generated fuzzy supra topologies, and several relations between different fuzzy ideals and fuzzy supra topologies are also studied here.

In Section 5, we introduce a fuzzy set operator Ψ_S and study its properties.

In Section 6, we introduce some sets of fuzzy ideal supra topological spaces (fuzzy $*$-supra dense-in-itself sets, fuzzy S^*-supra closedsets, fuzzy $*$-supra perfect sets, fuzzy regular-I-supra closedsets, fuzzy-I-supra opensets, fuzzy semi-I-supra opensets, fuzzy pre-I-supra opensets, fuzzy α-I-supra opensets, and fuzzy β-I-supra opensets) and study some characteristics of theses sets. Finally, in Section 7, we introduce some fuzzy ideal supra continuous functions.

2. Preliminaries

Definition 1 (see [1]). Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function $\mu_A: X \longrightarrow [0,1]$, and $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy set A, for each $x \in X$. It is clear that A is completely determined by the set of tuples $A = \{\langle x, \mu_A(x)\rangle: x \in X\}$.

Definition 2 (see [1]). Let A and B be fuzzy sets of the form$A = \{\langle x, \mu_A(x)\rangle: x \in X\}$ and $B = \{\langle x, \mu_B(x)\rangle: x \in X\}$. Then, we define

(1) $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ for all $x \in X$

(2) $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$

(3) $C = A \cup B \Leftrightarrow \mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}$

(4) $D = A \cap B \Leftrightarrow \mu_D(x) = \min\{\mu_A(x), \mu_B(x)\}$

(5) $E = A^c \Leftrightarrow \mu_E(x) = 1 - \mu_A(x)$

(6) $1_x = X = \{\langle x, 1\rangle: x \in X\}$ and $0_x = \varnothing = \{\langle x, 0\rangle: x \in X\}$

Definition 3 (see [1]). For any family $\{A_j\}_{j\in J}$ of fuzzy sets in X, we define

(1) $C = \cup_{j\in J} A_j \Leftrightarrow \mu_C(x) = \sup\left\{\mu_{A_j}(x): j \in J\right\}$

(2) $D = \cap_{j\in J} A_j \Leftrightarrow \mu_D(x) = \inf\left\{\mu_{A_j}(x): j \in J\right\}$

Definition 4 (see [21]). A fuzzy set in X is called a fuzzy point if and only if it takes the value 0 for all $y \in X$ except one, say $x \in X$. If its value at x is $\alpha \in (0,1]$, we denote this fuzzy point by x_α, where the point x is called its support.

Definition 5 (see [18]). A subclass $S \subseteq P(x)$ ($P(X)$ is the collection of all fuzzy sets on X and is called a fuzzy supra topology on X if $0_x, 1_x \in S$, and S is closed under arbitrary union. The pair (X, S) is called a fuzzy supra topological space, and the members of S are called fuzzy supra opensets. A fuzzy set A is fuzzy supra closed if and only if its complement A^c is fuzzy supra open.

Definition 6 (see [18]). Let (X, S) be a fuzzy supra topological space and let A be a fuzzy set in X. Then, the fuzzy supra interior and the fuzzy supra closure of A in (X, S) are defined as

$$\begin{aligned} \mathrm{Int}^S(A) &= \bigcup\{U: U \subseteq A, U \in S\}, \\ \mathrm{Cl}^S(A) &= \cap\{F: A \subseteq F, F^c \in S\}, \end{aligned} \tag{1}$$

respectively.

Corollary 1. *From Definition 6, $Int^S(A)$ is a fuzzy supra openset and $Cl^S(A)$ is a fuzzy supra closedset.*

Definition 7 (see [22]). Let (X, S) be a fuzzy supra topological space. A fuzzy set A in X is said to be quasi-coincident with a fuzzy set B if there exists $x \in X$, such that $A(x) + B(x) > 1$. In this case, we write AqB.

Definition 8 (see [22]). A fuzzy set A in a fuzzy supra topological space (X, S) is an s-neighbourhood of a fuzzy point x_α if there is $M \in S$ with $x_\alpha \in M \subseteq A$. The collection $N^S(x_\alpha)$ of all s-neighbourhoods of x_α is called the s-neighbourhood system of x_α.

Definition 9 (see [18]). Let S_1 and S_2 be two fuzzy supra topologies on a set X such that $S_1 \subseteq S_2$. Then, we say that S_2 is stronger(finer) than S_1 or S_1 is weaker(coarser) than S_2.

Definition 10 (see [18]). Let (X, S) be a fuzzy supra topological space and $\beta \subseteq S$. Then, β is called a base for the fuzzy supra topology S if every fuzzy supra openset $U \in S$ is a union of members of β. Equivalently, β is a fuzzy supra base for S if, for any fuzzy point $x_\alpha \in U$, there exists $B \in \beta$ with $x_\alpha \in B \subseteq U$.

Definition 11 (see [18]). A mapping c: $P(X) \longrightarrow P(X)$ is said to be a fuzzy supra closure operator if it satisfies the following axioms:

(1) $c(0_x) = 0_x$

(2) $A \subseteq c(A)$ for every fuzzy set A in X

(3) $c(A) \cup c(B) \subseteq c(A \cup B)$ for every fuzzy sets A and B in X

(4) $c(c(A)) = c(A)$ for every fuzzy set A in X

Theorem 1 *(see [18]). Let X be a nonempty set, and let the mapping c: $P(X) \longrightarrow P(X)$ be a fuzzy supra closure operator. Then, the collection $S = \{A \in P(X): c(A^c) = A^c\}$ is fuzzy supra topology on X induced by the fuzzy supra closure operator c.*

Definition 12 (see [23]). A nonempty collection of fuzzy sets I of a set X is called a fuzzy ideal on X if and only if

(1) $A \in I$ and $B \subseteq A \Rightarrow B \in I$(heredity)

(2) $A \in I$ and $B \in I \Rightarrow A \cup B \in I$(finite additivity)

3. Fuzzy *S*-Local Function

Definition 13. A fuzzy supra topological space (X, S) with a fuzzy ideal I on X is called a fuzzy ideal supra topological space and denoted as (X, S, I).

Definition 14. Let (X, S, I) be a fuzzy ideal supra topology and let A be any fuzzy set in X. Then, the fuzzy s-local function $A^{*S}(\mathrm{I}, \mathrm{S})$ of A is the union of all fuzzy points x_α such that if $M \in N^S(x_\alpha)$ and $E \in I$, then there is at least one $y \in X$ for which $M(y) + A(y) - 1_x > E(y)$.

In other words, we say that a fuzzy set A is fuzzy s-local in I at x_α if there exists $M \in N^S(x_\alpha)$, such that for every $y \in X$, $M(y) + A(y) - 1_x \leq E(y)$ for some $E \in I$. We will occasionally write A^{*S} for $A^{*S}(I, S)$, and it will cause no ambiguity.

Example 1. The simplest fuzzy ideals on X are $\{0_x\}$ and $P(X)$. Then, $I = \{0_x\} \Leftrightarrow A^{*S} = \mathrm{Cl}^S(A)$, for any fuzzy set A in X and $I = P(X) \Leftrightarrow A^{*S} = 0_x$.

Theorem 2. *Let (X, S, I) be a fuzzy ideal supra topological space, and let A and B be fuzzy sets in X. Then,*

(1) $0_x^{*S} = 0_x$

(2) If $A \subseteq B$, *then* $A^{*S} \subseteq B^{*S}$

(3) If $I_1 \subseteq I_2$, *then* $A^{*S}(I_2) \subseteq A^{*S}(I_1)$

(4) $A^{*S} = Cl^S(A^{*S}) \subseteq Cl^S(A)$

(5) $(A^{*S})^{*S} \subseteq A^{*S}$

(6) A^{*S} *is a fuzzy supra closedset*

(7) $A^{*S} \cup B^{*S} \subseteq (A \cup B)^{*S}$

(8) $(A \cap B)^{*S} \subseteq A^{*S} \cap B^{*S}$

(9) If $E \in I$, *then* $(A \cup E)^{*S} = A^{*S} = (A - E)^{*S}$

(10) If $U \in S$, *then* $U \cap A^{*S} = U \cap (U \cap A)^{*S} \subseteq (U \cap A)^{*S}$

(11) If $E \in I$, *then* $E^{*S} = 0_x$

(12) If $E \in I$, *then* $(1_x - E)^{*S} = 1_x^{*S}$

Proof

(1) This is obvious from the definition of fuzzy s-local function

(2) Let $A \subseteq B$. Then, $A(x) \leq B(x)$ for every $x \in X$. From the definition of fuzzy s-local function, if $x_\alpha \in A^{*S}$, then $x_\alpha \in B^{*S}$. Therefore, $A^{*S} \subseteq B^{*S}$.

(3) Let $I_1 \subseteq I_2$ and let $x_\alpha \in X$ be any fuzzy points, such that $x_\alpha \notin A^{*S}(I_1, S)$. Then, there is at least one $M \in N^S(x_\alpha)$ and for every $y \in X$, such that $M(y) + A(y) - 1_x \leq E(y)$ for some $E \in I_1$. But $I_1 \subseteq I_2$, then $E \in I_2$. This implies $x_\alpha \notin A^{*S}(I_2, S)$. Therefore, $A^{*S}(I_2, S) \subseteq A^{*S}(I_1, S)$.

(4) For any fuzzy ideal on X, $\{0_x\} \subseteq I$. Therefore by (3) and Example 1, for any fuzzy set A in $X, A^{*S}(I) \subseteq A^{*S}(\{0_x\}) = \mathrm{Cl}^S(A)$. Now, let $x_\alpha \in \mathrm{Cl}^S(A^{*S})$. Then, for every $M \in N^S(x_\alpha)$, there is at least one $y \in X$, such that $M(y) + A^{*S}(y) > 1_x$. Hence, $A^{*S}(y) \neq 0_x$, and let $\beta = A^{*S}(y)$. Clearly, $y_\beta \in A^{*S}$ and $\beta + M(y) > 1_x$, so that $M \in N^S(y_\beta)$. Now, $y_\beta \in A^{*S}$ implies there is at least one $x_1 \in X$, such that $V(x_1) + A(x_1) - 1_x > E(x_1)$, for each $V \in N^S(y_\beta)$ and $E \in I$. This is also true for M. So, there is at least one $x_2 \in X$ such that $M(x_2) + A(x_2) - 1_x > E(x_2)$, for each $E \in I$. Since $M \in N^S(x_\alpha)$; therefore, $x_\alpha \in A^{*S}$. Hence, $A^{*S} = \mathrm{Cl}^S(A^{*S}) \subseteq \mathrm{Cl}^S(A)$.

(5) From (4), $(A^{*S})^{*S} \subseteq \mathrm{Cl}^S(A^{*S}) = A^{*S}$

(6) Let $x_\alpha \notin A^{*S}(I, S)$. Then, there is at least one $M \in N^S(x_\alpha)$, such that for every $y \in X, M(y) + A(y) - 1_x \leq E(y)$ for some $E \in I$, and $x_\alpha \in M$ implies $M \leq 1_x - A^{*S}(I)$, and we have $1_x - A^{*S}(I)$ which is a fuzzy supra openset. Therefore, A^{*S} is fuzzy supra closed.

(7) We have $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Then, from (2), $A^{*S} \subseteq (A \cup B)^{*S}$ and $B^{*S} \subseteq (A \cup B)^{*S}$. Hence, $A^{*S} \cup B^{*S} \subseteq (A \cup B)^{*S}$.

(8) We have $(A \cap B) \subseteq A$ and $(A \cap B) \subseteq B$. Then, from (2), $(A \cap B)^{*S} \subseteq A^{*S}$ and $(A \cap B)^{*S} \subseteq B^{*S}$. Hence, $(A \cap B)^{*S} \subseteq A^{*S} \cap B^{*S}$.

(9) We have $A \subseteq A \cup E$. Then, from (2), $A^{*S} \subseteq (A \cup E)^{*S}$. Now, let $x_\alpha \in (A \cup E)^{*S}$; then, for every $M \in N^S(x_\alpha)$, there is at least one $y \in X$ such that $M(y) + (A \cup E)(y) - 1_x > E(y)$, for each $E \in I$. If $\max\{A(y), E(y)\} = E(y)$, then for each $M \in N^S(x_\alpha)$ and $E \in I$, $M(y) + E(y) - 1_x > E(y)$, and this implies $M(y) > 1_x$, and this is a contradiction. Then, for each $M \in N^S(x_\alpha)$ and $E \in I, M(y) + A(y) - 1_x > E(y)$, and this implies $x_\alpha \in A^{*S}$. Therefore, $(A \cup E)^{*S} \subseteq A^{*S}$. Hence, $(A \cup E)^{*S} = A^{*S}$. Now, from the defined fuzzy s-

local function, clear $A^{*S} = (A - E)^{*S}$; then, $(A \cup E)^{*S} = A^{*S} = (A - E)^{*S}$.

(10) We have $U \cap A \subseteq A$. Then, from (2), $(U \cap A)^{*S} \subseteq A^{*S}$. So, $U \cap (U \cap A)^{*S} \subseteq U \cap A^{*S}$.

(11) This is obvious from the definition of fuzzy s-local function

(12) This is obvious from (9) □

Theorem 3. *Let (X, S, I) be a fuzzy ideal supra topological space, and let A be any fuzzy set in X. Then,* $(A \cup A^{*S})^{*S} \subseteq A^{*S}$.

Proof.

Let $x_\alpha \in (A \cup A^{*S})^{*S}$. Then, for every $M \in N^S(x_\alpha)$, there is at least one $y \in X$, such that $M(y) + (A \cup A^{*S})(y) - 1_x > E(y)$, for each $E \in I$. Now, if $\max\{A(y), A^{*S}(y)\} = A(y)$ implies $x_\alpha \in A^{*S}$; therefore, the proof is done. If $\max\{A(y), A^{*S}(y)\} = A^{*S}(y)$, then $A^{*S}(y) \neq 0_x$, and let $\beta = A^{*S}$. Clearly, $y_\beta \in A^{*S}$ and $M(y) + \beta - 1_x > E(y)$. Now, $y_\beta \in A^{*S}$ implies there is at least one $x_1 \in X$, such that $V(x_1) + A(x_1) - 1_x > E(x_1)$ for each $V \in N^S(y_\beta)$ and $E \in I$. This is also true for M. So, there is at least one $x_2 \in X$, such that $M(x_2) + A(x_2) - 1_x > E(x_2)$, for each $E \in I$. Since $M \in N^S(x_\alpha)$, then $x_\alpha \in A^{*S}$. Therefore, $(A \cup A^{*S})^{*S} \subseteq A^{*S}$. □

Theorem 4. *Let (X, S, I) be a fuzzy ideal supra topological space. Then, the operator* $Cl^{*S}: P(X) \longrightarrow P(X)$, *defined by* $Cl^{*S}(A) = A \cup A^{*S}$ *for any fuzzy set A in X, is a fuzzy supra closure operator, and hence, it generates a fuzzy supra topology* $S^*(I) = \{A \in P(X): Cl^{*S}(A^c) = A^c\}$ *which is finer than S.*

Proof (1) By (1) in Theorem 2, $0_x^{*S} = 0_x$, we have $\mathrm{Cl}^{*S}(0_x) = 0_x$

(2) Clear that $A \subseteq \mathrm{Cl}^{*S}(A)$ for every fuzzy set A

(3) Let A and B be any two fuzzy sets. Then, $\mathrm{Cl}^{*S}(A) \cup \mathrm{Cl}^{*S}(B) = (A \cup A^{*S}) \cup (B \cup B^{*S}) = (A \cup B) \cup (A^{*S} \cup B^{*S}) \subseteq (A \cup B) \cup (A \cup B)^{*S} = \mathrm{Cl}^{*S}(A \cup B)$ (by (7) in Theorem 2). Hence, $\mathrm{Cl}^{*S}(A) \cup \mathrm{Cl}^{*S}(B) \subseteq \mathrm{Cl}^{*S}(A \cup B)$.

(4) Let A be any fuzzy set. Since, by (2), $A \subseteq \mathrm{Cl}^{*S}(A)$; then, $\mathrm{Cl}^{*S}(A) \subseteq \mathrm{Cl}^{*S}(\mathrm{Cl}^{*S}(A))$. On the other hand, $\mathrm{Cl}^{*S}(\mathrm{Cl}^{*S}(A)) = \mathrm{Cl}^{*S}(A \cup A^{*S}) = (A \cup A^{*S}) \cup (A \cup A^{*S})^{*S} \subseteq A \cup A^{*S} \cup A^{*S} = \mathrm{Cl}^{*S}(A)$ (by Theorem 3); it follows that $\mathrm{Cl}^{*S}(\mathrm{Cl}^{*S}(A)) \subseteq \mathrm{Cl}^{*S}(A)$. Hence, $\mathrm{Cl}^{*S}(\mathrm{Cl}^{*S}(A)) = \mathrm{Cl}^{*S}(A)$. Consequently, $\mathrm{Cl}^{*S}(A)$ is a fuzzy supra closure operator. Also, it is easy to show that the collection $S^*(I) = \{A \in P(X): \mathrm{Cl}^{*S}(A^c) = A^c\}$ is a fuzzy supra topology on X which is called the fuzzy supra topology induced by the fuzzy supra closure operator. □

Proposition 1. *For any fuzzy ideal on X, if* $I = \{0_x\} \Rightarrow Cl^{*S}(A) = A \cup A^{*S} = A \cup Cl^S(A) = Cl^S(A)$ *for every a fuzzy set A in X. So,* $S^*(\{0_x\}) = S$, *and if* $I = P(X) \Rightarrow Cl^{*S}(A) = A$ *because* $A^{*S} = 0_x$ *for every a fuzzy set A in X. So,* $S^*(P(X))$ *is a fuzzy discrete supra topology on X. Since* $\{0_x\}$ *and P(X) are the tow extreme fuzzy ideal on X, for any fuzzy ideal I on X, we have* $\{0_x\} \subseteq I \subseteq P(X)$. *So, we can conclude by (2) in Theorem 2 that* $S^*(\{0_x\}) \subseteq S^*(I) \subseteq S^*(P(X))$, *i.e.,* $S \subseteq S^*(I)$, *for any fuzzy ideal I on X. In particular, we have for any tow fuzzy ideals* I_1 *and* I_2 *on X,* $I_1 \subseteq I_2 \Rightarrow S^*(I_1) \subseteq S^*(I_2)$.

Theorem 5. *For any fuzzy ideal supra topological space (X, S, I), the class* $\beta(I, S) = \{U - E: U \in S, E \in I\}$ *is the base for the fuzzy supra topology* $S^*(I)$.

Proof.

Since $0_x \in I$, then $S \subseteq \beta$ from which it follows $X = \bigcup \beta$. Also, for every $\beta_1, \beta_2 \in \beta$, we have, $\beta_1 = U_1 - E_1$ and $\beta_2 = U_2 - E_2$, where $U_1, U_2 \in S^*(I)$, and $E_1, E_2 \in I$. Then, $\beta_1 \cap \beta_2 = (U_1 - E_1) \cap (U_2 - E_2) = (U_1 \cap E_1^c) \cap (U_1 \cap E_2^c) = (U_1 - U_2) \cap (E_1 - E_2)^c = (U_1 - U_2) - (E_1 \cup E_2)^c \in \beta$. Therefore, β is a base for $S^*(I)$. □

Example 2. Let T be the fuzzy indiscrete supra topology on X, i.e., $T = \{0_x, 1_x\}$. So, 1_x is the only s-neighbourhoods of x_α. Now, $x_\alpha \in A^{*S}$ for a fuzzy set A if and only if for each $E \in I$, there is at least one $y \in X$, such that $1_x + A(y) - 1_x > E(y)$. This implies, for each $E \in I, A(y) > E(y)$ for at least one $y \in X$. So, $A \notin I$. Therefore, $A^{*S} = 1_x$ if $A \notin I$ and $A^{*S} = 0_x$ if $A \in I$. This implies that we have $\mathrm{Cl}^{*S}(A) = A \cup A^{*S} = 1_x$ if $A \notin I$ and $\mathrm{Cl}^{*S}(A) = A$ if $A \in I$ for any fuzzy set A of X. Hence, $T^* = \{M: M^c \in I\}$. Let $S \cup T^*(I)$ be the supremum fuzzy supra topology of S and $T^*(I)$, i.e., the smallest fuzzy supra topology is generated by $S \cup T^*(I)$. Then, we have the following theorem.

Theorem 6. $S^*(I) = S \cup T^*(I)$.

Proof. Follows from the fact that β forms a basis for $S^*(I)$. □

Theorem 7. *Let* $S_1 and S_2$ *be tow fuzzy supra topologies on X. Then, for any fuzzy ideal I on X,* $S_1 \subseteq S_2$ *implies*

(1) $A^{*S}(S_2, I) \subseteq A^{*S}(S_1, I)$ *for every fuzzy set in X*

(2) $S_1^*(I) \subseteq S_2^*(I)$

Proof.

(1) Since every $S_1 s$-neighbourhood of any fuzzy point x_α is also a $S_2 s$-neighbourhood of x_α. Therefore, $A^{*S}(S_2, I) \subseteq A^{*S}(S_1, I)$.

(2) Clearly, $S_1^*(I) \subseteq S_2^*(I)$ as $A^{*S}(S_2, I) \subseteq A^{*S}(S_1, I)$ □

Theorem 8. *Let (X, S, I) be a fuzzy supra topological space. Then,* *(1) If $H \in I \Rightarrow H^c \in S^*(I)$*

(2) $A^{*S} = Cl^{*S}(A^{*S})$ for every fuzzy set A in X, i.e., A^{*S} is a fuzzy S^*-supra closedset.

Proof.

(1) For every $H \in I \Rightarrow H^{*S} = 0_x$. Hence, $\mathrm{Cl}^{*S}(H) = H \Rightarrow H^c \in S^*(I)$, i.e., H is a fuzzy S^*-supra closedset.

(2) From (5) in Theorem 2, we have $(A^{*S})^{*S} \subseteq A^{*S} \Rightarrow A^{*S} = A^{*S} \cup (A^{*S})^{*S} = \mathrm{Cl}^{*S}(A)$. Hence, A^{*S} is a fuzzy S^*-supra closedset. □

4. *S*-Compatible of Fuzzy Ideals with Fuzzy Supra Topology

Definition 15. Let (X, S, I) be a fuzzy supra topological space. S is said to be fuzzy S-compatible with I, denoted by $S \sim I$, if for every fuzzy set A in X; if for all fuzzy point $x_\alpha \in A$, there exists $M \in N^S(x_\alpha)$ such that $M(y) + A(y) - 1_x \le E(y)$ holds for every $y \in X$ and some $E \in I$, and then, $A \in I$.

Definition 16. (see [20]). Let $\{B_\alpha, \alpha \in \Delta\}$ be any indexed family of fuzzy sets in X such that $B_\alpha q A$, for each $\alpha \in \Delta$, where A is a fuzzy set in X. Then, $\{B_\alpha, \alpha \in \Delta\}$ is said to be a quasicover of A if and only if $A(y) + \cup_{\alpha\in\Delta} B_\alpha(y) \ge 1_x$ for every $y \in X$.

Definition 17. Let $\{B_\alpha, \alpha \in \Delta\}$ be quasicover of A; if each B_α is a fuzzy supra openset, then this quasicover with be called a fuzzy quasi-supra open cover A in X. Therefore, in either case, $A^c \subseteq \cup_{\alpha\in\Delta} B_\alpha$.

Theorem 9. *Let (X, S, I) be a fuzzy ideal supra topological space. Then, the following conditions are equivalent:*

(1) $S \sim I$

(2) If for every fuzzy set A in X has a fuzzy quasi-supra open cover $\{B_\alpha, \alpha \in \Delta\}$ such that for each α, $A(y) + B_\alpha(y) - 1_x \le E(y)$ for some $E \in I$ and for every $y \in X$, and then, $A \in I$

(3) For every fuzzy set A in X, $A \cap A^{*S} = 0_x$ implies $A \in I$

(4) For every fuzzy set A in X, $\widetilde{A} \in I$, where $\widetilde{A} = \bigcup x_\alpha$, such that $x_\alpha \in A$ but $x_\alpha \notin A^{*S}$

Proof. (1)⇒(2) Let $\{B_\alpha, \alpha \in \Delta\}$ be a fuzzy quasi-supra open cover of a fuzzy set A in X such that for each $\alpha \in \Delta$, $B_\alpha(y) + A(y) - 1_x \le E(y)$ for some $E \in I$ and for every $y \in X$. Therefore, as $\{B_\alpha, \alpha \in \Delta\}$ is a fuzzy quasi-supra open cover of A, for each $x_\alpha \in A$, there exists at least one B_{α_0} such that $x_\alpha q B_{\alpha_0}$ and for every $y \in X$, $B_{\alpha_0}(y) + A(y) - 1_x \le E(y)$ for some $E \in I$. Obviously, $B_{\alpha_0} \in N^S(x_\alpha)$. Therefore, as $S \sim I$, $A \in I$.

(2)⇒(1) Clear from the fact that a collection of fuzzy supra opensets which contain at least one open s-neighbourhood of each fuzzy point of A that constitutes a fuzzy quasi-supra open cover of A.

(2)⇒(3) Let $A \cap A^{*S} = 0_x$, i.e., $\min\{A(y), A^{*S}(y)\} = 0_x$, for every $y \in X$. So, a fuzzy point $x_\alpha \in A$ implies $x_\alpha \notin A^{*S}$. That means, there is $M \in N^S(x_\alpha)$ such that for every $y \in X$, $M(y) + A(y) - 1_x \le E(y)$ for some $E \in I$. If $x_\alpha \in A$ since $M \in N^S(x_\alpha)$, there is a fuzzy supra openset V (in S) such that $x_\alpha q V \subseteq M$, and so, the collection of such V's for each $x_\alpha \in A$ constitutes a fuzzy quasi-supra open cover of A. Therefore, by condition (2), $A \in I$.

(3)⇒(1) Let for every fuzzy point $x_\alpha \in A$, there is $M \in N^S(x_\alpha)$ such that for every $y \in X$, $M(y) + A(y) - 1_x \le E(y)$ for some $E \in I$. That means $x_\alpha \notin A^{*S}$. Now, there are two cases: either $A^{*S}(x) = 0_x$ or $A^{*S}(x) \ne 0_x$, but $\alpha > A^{*S}(x) \ne 0_x$. Let, if possible, $x_\alpha \in A$ be such that $\alpha > A^{*S}(x) \ne 0_x$. Let $\alpha_1 = A^{*S}(x)$. Then, the fuzzy point $x_{\alpha_1} \in A^{*S}$ and also $x_{\alpha_1} \in A$. This implies for each $V \in N^S(x_{\alpha_1})$, and for each $E \in I$, there is at least one $y \in X$ such that $V(y) + A(y) - 1_x > E(y)$. Since $x_{\alpha_1} \in A$, this contradicts the assumption for every fuzzy point of A. So, $A^{*S}(x) = 0_x$. That means, $x_\alpha \in A$ implies $x_\alpha \notin A^{*S}$. Now, this is true for every fuzzy set A in X. So, for every fuzzy set A in X, $A \cap A^{*S} = 0_x$. Hence, by condition (3), we have $A \in I$, which implies $S \sim I$.

(3)⇒(4) Let the fuzzy point $x_\alpha \in \widetilde{A}$. That means, $x_\alpha \in A$, but $x_\alpha \notin A^{*S}$. So, there is $M \in N^S(x_\alpha)$ such that for every $y \in X$, $M(y) + A(y) - 1_x \le E(y)$ for some $E \in I$. Since $\widetilde{A} \subseteq A$, so for every $y \in X$, $M(y) + \widetilde{A}(y) - 1_x \le E(y)$ for some $E \in I$. Therefore, $x_\alpha \notin \widetilde{A}^{*S}$, so that either $\widetilde{A}^{*S}(x) = 0_x$ or $\widetilde{A}^{*S}(x) \ne 0_x$, but $\alpha > \widetilde{A}^{*S}(x) \ne 0_x$. Let x_{α_1}, be a fuzzy point such that $\alpha_1 \le \widetilde{A}^{*S}(x) < \alpha$, i.e., $x_{\alpha_1} \in \widetilde{A}^{*S}$. So, for each $V \in N^S(x_{\alpha_1})$ and for each $E \in I$, there is at least one $y \in X$ such that $V(y) + \widetilde{A}(y) - 1_x > E(y)$. Since $\widetilde{A} \subseteq A$, for each $V \in N^S(x_{\alpha_1})$ and for each $E \in I$, there is at least one $y \in X$ such that $V(y) + A(y) - 1_x > E(y)$. This implies $x_{\alpha_1} \in A^{*S}$. But as $\alpha_1 < \alpha$, $x_\alpha \in \widetilde{A}$, and therefore $x_{\alpha_1} \notin A^{*S}$. This is a contradiction. Hence, $\widetilde{A}^{*S}(x) = 0_x$, so that $x_\alpha \in \widetilde{A}$ implies $x_\alpha \notin \widetilde{A}^{*S}$ with $\widetilde{A}^{*S} = 0_x$. Thus, we have $\widetilde{A} \cap \widetilde{A}^{*S} = 0_x$ for every fuzzy set A in X. Hence, by condition (3), $\widetilde{A} \in I$.

(4) ⇒(1) Let for every fuzzy set A in X, $\widetilde{A} \in I$. This means for every $x_\alpha \in \widetilde{A}$ then $x_\alpha \in A$ and $x_\alpha \notin A^{*S}$ then for every $y \in X$, there exist $M \in N^S(x_\alpha)$ such that $M(y) + A(y) - 1_x \le E(y)$ for some $E \in I$. This implies $S \sim I$. □

Theorem 10. *Let (X, S, I) be a fuzzy ideal supra topological space. Then, the following are equivalent and implied by $S \sim I$.*

(1) For every fuzzy set A in X, $A \cap A^{*S} = 0_x$ implies $A^{*S} = 0_x$.

(2) *For every fuzzy set A in X,* $\widetilde{A}^{*S} = 0_x$ *($\tilde{A}$ is defined as in (4) in Theorem 9).*

(3) *For every fuzzy set A in X,* $(A \cap A^{*S})^{*S} = A^{*S}$.

Proof. Clear from Theorem 9. □

Theorem 11. *Let (X, S, I) be a fuzzy ideal supra topological space, let* $S \sim I$. *Then a fuzzy set A in X is supra closed with respect to* $S^*(I)$ *if and only if it is the union of a fuzzy set which is supra closed with respect to S and a fuzzy set in I.*

Proof. Let A be a fuzzy set in X such that it is fuzzy S^*-supra closedset. That means $A^{*S} \subseteq A$, and we have $A = \widetilde{A} \cup A^{*S}$. Since $S \sim I$, $A \in I$. Also, A^{*S} is always fuzzy S-supra closedset by (3) in Theorem 2. Conversely, let A be any fuzzy set in X such that $A = U \cup V$, where $\mathrm{Cl}^{*S}(U) = U \subseteq A$. This means $A^{*S} \subseteq U \subseteq A$. So, we have $\mathrm{Cl}^{*S}(A) = A \cup A^{*S} = A$ and implies A is a fuzzy S^*-supra closedset. □

Corollary 2. *The fuzzy supra topology S is S-compatible with fuzzy ideal I on X implies* $\beta(I, S)$*, a basis for* $S^*(I)$*, and is itself a fuzzy topology and also* $\beta = S^*(I)$.

Proof. Clear. □

5. Fuzzy Set Operator Ψ_S

In this section, we introduce the fuzzy set operator Ψ_S, S-codense, and Ψ_S-C-fuzzy set, and we give new results.

Definition 18. Let (X, S, I) be a fuzzy ideal supra topological space. An operator Ψ_S: $P(X) \longrightarrow S$ is defined as follows: for every fuzzy set A in X, $\Psi_S(A) = \{x_\alpha$ fuzzy point: there exists $M \in N^S(x_\alpha)$ such that $M - A \in I\}$. We observe that $\Psi_S(A) = 1_x - (1_x - A)^{*S}$. The behaviors of the operator Ψ_S have been discussed in the following theorem:

Theorem 12. *Let (X, S, I) be a fuzzy ideal supra topological space. Let A and B be two fuzzy sets in X. Then,*

(1) $\Psi_S(A)$ *is a fuzzy supra openset*

(2) $\mathrm{Int}^S(A) \subseteq \Psi_S(A)$

(3) *If* $A \subseteq B$, *then* $\Psi_S(A) \subseteq \Psi_S(B)$

(4) $\Psi_S(A \cap B) \subseteq \Psi_S(A) \cap \Psi_S(B)$

(5) $\Psi_S(A) \cup \Psi_S(B) \subseteq \Psi_S(A \cup B)$

(6) *If* $U \in S$, *then* $U \subseteq \Psi_S(U)$

(7) $\Psi_S(A) \subseteq \Psi_S(\Psi_S(A))$

(8) $\Psi_S(A) = \Psi_S(\Psi_S(A))$ *if and only if* $(1_x - A)^{*S} = ((1_x - A)^{*S})^{*S}$

(9) *If* $(A - B) \cup (B - A) \in I$, *then* $\Psi_S(A) = \Psi_S(B)$

(10) *If* $E \in I$, *then* $\Psi_S(E) = 1_x - 1_x^{*S}$

(11) *If* $E \in I$, *then* $\Psi_S(A - E) = \Psi_S(A)$

(12) *If* $E \in I$, *then* $\Psi_S(A \cup E) = \Psi_S(A)$

Proof. (1) Since $(1_x - A)^{*S}$ is a fuzzy supra closedset, $1_x - (1_x - A)^{*S}$ is a fuzzy supra openset. Hence, $\Psi_S(A)$ is a fuzzy supra openset.

(2) From definition of the Ψ_S operator, $\Psi_S(A) = 1_x - (1_x - A)^{*S}$. Then, $1_x - \mathrm{Cl}^S(1_x - A) \subseteq 1_x - (1_x - A)^{*S} = \Psi_S(A)$, from (4) in Theorem 2. Hence, $\mathrm{Int}^S(A) \subseteq \Psi_S(A)$.

(3) Let $A \subseteq B$. Then, $(1_x - B) \subseteq (1_x - A)$. Then, from (2) in Theorem 2, $(1_x - B)^{*S} \subseteq (1_x - A)^{*S}$. Therefore, $\Psi_S(A) \subseteq \Psi_S(B)$.

(4) We have $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Then, from (3), $\Psi_S(A \cap B) \subseteq \Psi_S(A) \cap \Psi_S(B)$.

(5) We have $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Then, from (3), $\Psi_S(A) \cup \Psi_S(B) \subseteq \Psi_S(A \cup B)$.

(6) Let $U \in S$. Then, $(1_x - U)$ is a fuzzy supra closedset, and hence, $\mathrm{Cl}^S(1_x - U) = (1_x - U)$. Then, $(1_x - U)^{*S} \subseteq \mathrm{Cl}^S(1_x - U) = (1_x - U)$. Hence, $U \subseteq 1_x - (1_x - U)^{*S}$, so $U \subseteq \Psi_S(U)$.

(7) From (2), $\Psi_S(A) \in S$, and from (6), $\Psi_S(A) \subseteq \Psi_S(\Psi_S(A))$.

(8) Let $\Psi_S(A) = \Psi_S(\Psi_S(A))$. Then, $1_x - (1_x - A)^{*S} = \Psi_S(1_x - (1_x - A)^{*S}) = 1_x - (1_x - (1_x - (1_x - A)^{*S})^{*S}) = 1_x - ((1_x - A)^{*S})^{*S}$. Therefore, $(1_x - A)^{*S} = ((1_x - A)^{*S})^{*S}$. Conversely suppose that $(1_x - A)^{*S} = ((1_x - A)^{*S})^{*S}$ hold. Then, $1_x - (1_x - A)^{*S} = 1_x - ((1_x - A)^{*S})^{*S}$ and $1_x - (1_x - A)^{*S} = 1_x - (1_x - (1_x - (1_x - A)^{*S}))^{*S} = 1_x - (1_x - \Psi_S(A))^{*S}$. Hence, $\Psi_S(A) = \Psi_S(\Psi_S(A))$.

(9) Let $(A - B) \cup (B - A) \in I$, and let $A - B = E_1, B - A = E_2$. We observe that $E_1, E_2 \in I$ by heredity, and $B = (A - E_1) \cup E_2$. Thus, $\Psi_S(A) = \Psi_S(A - E_1) = \Psi_S((A - E_1) \cup E_2) = \Psi_S(B)$.

(10) By (9) in Theorem 2, we obtain if $E \in I$, then $\Psi_S(E) = 1_x - 1_x^{*S}$

(11) This follows from (9) in Theorem 2, and $\Psi_S(A - E) = 1_x - (1_x - (A - E))^{*S} = 1_x - ((1_x - A) \cup E)^{*S} = 1_x - (1_x - A)^{*S} = \Psi_S(A)$

(12) This follows from (9) in Theorem 2, and $\Psi_S(A \cup E) = 1_x - (1_x - (A \cup E))^{*S} = 1_x - ((1_x - A) - E)^{*S} = 1_x - (1_x - A)^{*S} = \Psi_S(A)$ □

Theorem 13. *Let (X, S, I) be a fuzzy ideal supra topological space. If* $\eta = \{A \in P(X)\colon A \subseteq \Psi_S(A)\}$. *Then,* η *is a fuzzy supra topology for X.*

Proof. Let $\eta = \{A \in P(X)\colon A \subseteq \Psi_S(A)\}$. By (1) in Theorem 2, $0_x^{*S} = 0_x$ and $\Psi_S(1_x) = 1_x - (1_x - 1_x)^{*S} = 1_x - 0_x^{*S} = 1_x$. Moreover, $\Psi_S(0_x) = 1_x - (1_x - 0_x)^{*S} = 1_x - 1_x = 0_x$. Therefore, we obtain that $0_x \subseteq \Psi_S(0_x)$ and $1_x \subseteq \Psi_S(1_x) = 1_x$, and thus, $0_x, 1_x \in \eta$. Now, if $\{A_\alpha\colon \alpha \in \Delta\} \subseteq \eta$, then $A_\alpha \subseteq \Psi_S(A_\alpha) \subseteq \Psi_S(\cup A_\alpha)$ for every α, and hence, $\bigcup A_\alpha \subseteq \Psi_S(\bigcup A_\alpha)$. This shows that η is a fuzzy supra topology. □

Definition 19. A fuzzy ideal I in a space (*X, S, I*) is called *S*-codense fuzzy ideal if $S \cap I = \{0_x\}$. The following theorem is related to *S*-codense fuzzy ideal.

Theorem 14. *Let (X, S, I) be a fuzzy ideal supra topological space, and let I be S-codense with S. Then,* $1_x = 1_x^{*S}$.

Proof. It is obvious that $1_x^{*S} \subseteq 1_x$. For converse, suppose $x_\alpha \in 1_x$ but $x_\alpha \notin 1_x^{*S}$, there is at least one $M \in N^S(x_\alpha)$ and for every $y \in X$ such that $M(y) + 1_x - 1_x \leq E(y)$ for some $E \in I \Rightarrow M(y) \leq E(y)$. That means $M \in I$, a contradiction to the fact that $S \cap I = \{0_x\}$. Hence, $1_x = 1_x^{*S}$. □

Definition 20. Let (*X, S, I*) be a fuzzy ideal supra topological space. A fuzzy set *A* in *X* is called the Ψ_S-*C*-fuzzy set if $A \subseteq \mathrm{Cl}^S(\Psi_S(A))$. The collection of all Ψ_S-*C*-fuzzy sets in (*X, S, I*) is denoted by $\Psi_S(X, S)$.

Theorem 15. *Let (X, S, I) be a fuzzy ideal supra topological space. If* $A \in S$, *then* $A \in \Psi_S(X, S)$.

Proof. From (6) in Theorem 12, it follows that $S \subseteq \Psi_S(X, S)$. □

Theorem 16. *Let* $\{A_\alpha: \alpha \in \Delta\}$ *be a collection of nonempty* Ψ_S-*C-fuzzy sets in a fuzzy ideal supra topological space (X, S, I); then,* $\cup_{\alpha\in\Delta} A_\alpha \in \Psi_S(X, S)$.

Proof. For each $\alpha \in \Delta$, $A_\alpha \subseteq \mathrm{Cl}^S(\Psi_S(A_\alpha)) \subseteq \mathrm{Cl}^S(\Psi_S(\cup_{\alpha\in\Delta} A_\alpha))$. This implies that $\cup_{\alpha\in\Delta} A_\alpha \subseteq Cl^S(\Psi_S(\cup_{\alpha\in\Delta} A_\alpha))$. Thus, $\cup_{\alpha\in\Delta} A_\alpha \in \Psi_S(X, S)$. □

6. Some Sets of a Fuzzy Ideal Supra Topological Space

Definition 21. Let (*X, S, I*) be a fuzzy ideal supra topological space and let *A* be any fuzzy set in *X*. Then, *A* is said to be

(1) Fuzzy $*$-supra dense-in-itself set if $A \subseteq A^{*S}$
(2) Fuzzy S^*-supra closedset if $A^{*S} \subseteq A$
(3) Fuzzy $*$-supra prefect set if $A = A^{*S}$
(4) Fuzzy regular-I-supra closedset if $A = (\mathrm{Int}^S(A))^{*S}$

Theorem 17. *Let (X, S, I) be a fuzzy ideal supra topological space, and let A be any fuzzy set in X. Then, the following statements hold:*

(1) Every fuzzy regular-I-supra closedset is a fuzzy $*$*-supra prefect set*
(2) Every fuzzy $*$*-supra prefect set is a fuzzy* S^**-supra closedset*
(3) Every fuzzy $*$*-supra prefect set is a fuzzy* $*$*-supra dense-in-itself set*

Proof. (1) Let *A* be a fuzzy regular-I-supra closedset. Then, we have $A = (\mathrm{Int}^S(A))^{*S}$. Since $\mathrm{Int}^S(A) \subseteq A$ by (2) in Theorem 2, then $(\mathrm{Int}^S(A))^{*S} \subseteq A^{*S}$. We have $A = (\mathrm{Int}^S(A))^{*S} \subseteq A^{*S}$. Since $A = (\mathrm{Int}^S(A))^{*S}$, then $A^{*S} = ((\mathrm{Int}^S(A))^{*S})^{*S} \subseteq (\mathrm{Int}^S(A))^{*S} = A$. Therefore, we obtain $A = A^{*S}$. This shows that *A* is a fuzzy $*$-supra prefect set.

(2) Let *A* be a fuzzy $*$-supra perfect set. Then, we have $A = A^{*S}$; therefore, we obtain $A^{*S} \subseteq A$. This shows that *A* is a fuzzy S^*-supra closedset.

(3) Let *A* be a fuzzy $*$-supra perfect set. Then, we have $A = A^{*S}$; therefore, we obtain $A \subseteq A^{*s}$. This shows that *A* is a fuzzy $*$-supra dense-in-itself set. □

Remark 1. The converses of Theorem 17 need not be true as the following examples show.

Example 3. Let $X = \{a, b, c\}$ and let *A* and *B* be fuzzy sets in *X* defined as follows:

$$\begin{aligned} A(a) &= 0.4, \\ A(b) &= 0.7, \\ A(c) &= 0.5, \\ B(a) &= 0.6, \\ B(b) &= 0.3, \\ B(c) &= 0.5. \end{aligned} \tag{2}$$

We put $S = \{0_x, 1_x, A\}$. If we take $I = \{0_x\}$, then *B* is a fuzzy $*$-supra perfect set but not a fuzzy regular-*I*-supra closedset.

Example 4. Let $X = \{a, b, c\}$, and let *A* and *B* be fuzzy sets in *X* defined as follows:

$$\begin{aligned} A(a) &= 0.1, \\ A(b) &= 0.3, \\ A(c) &= 0.5, \\ B(a) &= 0.4, \\ B(b) &= 0.6, \\ B(c) &= 0.7. \end{aligned} \tag{3}$$

We put $S = \{0_x, 1_x, A\}$. If we take $I = P(X)$, then *B* is a fuzzy S^*-supra closedset but not a fuzzy $*$-supra perfect set.

Example 5. In Example 3, *A* is a fuzzy $*$-supra dense-in-itself set but not a fuzzy $*$-supra perfect set.

Definition 22. Let (*X, S, I*) be a fuzzy ideal supra topological space, and let *A* be any fuzzy set in *X*. Then, *A* is said to be

(1) Fuzzy-I-supra openset if $A \subseteq \mathrm{Int}^S(A^{*S})$
(2) Fuzzy semi-I-supra openset if $A \subseteq \mathrm{Cl}^{*S}(\mathrm{Int}^S(A))$
(3) Fuzzy pre-I-supra openset if $A \subseteq \mathrm{Int}^S(\mathrm{Cl}^{*S}(A))$
(4) Fuzzy α-I-supra openset if $A \subseteq \mathrm{Int}^S(\mathrm{Cl}^{*S}(\mathrm{Int}^S(A)))$
(5) Fuzzy β-I-supra openset if $A \subseteq \mathrm{Cl}^S(\mathrm{Int}^S(\mathrm{Cl}^{*S}(A)))$

A fuzzy set A of a fuzzy ideal supra topological space (X, S, I) is said to be fuzzy-I-closedset (resp. fuzzy semi-I-supra closedset, fuzzy pre-I-supra closedset, fuzzy α-I-supra closedset, and fuzzy β-I-supra closedset) if its complement is a fuzzy-I-openset (resp. fuzzy semi-I-supra openset, fuzzy pre-I-supra openset, fuzzy α-I-supra openset, and fuzzy β-I-supra openset).

Theorem 18. *Let (X, S, I) be a fuzzy ideal supra topological space. Then, the following statements hold:*

(1) Every fuzzy supra openset is a fuzzy α-I-supra openset

(2) Every fuzzy-I-supra openset is a fuzzy pre-I-supra openset

(3) Every fuzzy α-I-supra openset is a fuzzy semi-I-supra openset

(4) Every fuzzy α-I-supra openset is a fuzzy pre-I-supra openset

(5) Every fuzzy semi-I-supra openset is a fuzzy β-I-supra openset

(6) Every fuzzy pre-I-supra openset is a fuzzy β-I-supra openset

Proof. (1) Let A be a fuzzy supra openset. Then, we have $A = \mathrm{Int}^S(A)$. Since, $A \subseteq \mathrm{Int}^S(A) \subseteq \mathrm{Cl}^{*S}(\mathrm{Int}^S(A))$. But if A is a fuzzy supra openset, then $A = \mathrm{Int}^S(A) \subseteq \mathrm{Int}^S(\mathrm{Cl}^{*S}(\mathrm{Int}^S(A)))$. This shows that A is a fuzzy α-I-supra openset.

(2) Let A be a fuzzy-I-supra openset. Then, we have $A \subseteq \mathrm{Int}^S(A^{*S})$, but $A^{*S} \subseteq \mathrm{Cl}^{*S}(A)$. Then, $A \subseteq \mathrm{Int}^S(\mathrm{Cl}^{*S}(A))$. This shows that A is a fuzzy pre-I-supra openset.

(3) Let A be a fuzzy α-I-supra openset. Then, we have $A \subseteq \mathrm{Int}^S(\mathrm{Cl}^{*S}(\mathrm{Int}^S(A)))\backslash\backslash \subseteq \mathrm{Cl}^{*S}(\mathrm{Int}^S(A))$. This shows that A is a fuzzy semi-I-supra openset.

(4) Let A be a fuzzy α-I-supra openset. Then, we have $A \subseteq \mathrm{Int}^S(\mathrm{Cl}^{*S}(\mathrm{Int}^S(A)))\backslash\backslash \subseteq \mathrm{Int}^S(\mathrm{Cl}^{*S}(A))$. This shows that A is a fuzzy pre-I-supra openset.

(5) Let A be a fuzzy semi-I-supra openset. Then, we have $A \subseteq \mathrm{Cl}^{*S}(\mathrm{Int}^S(A)) \subseteq \mathrm{Cl}^S(\mathrm{Int}^S(\mathrm{Cl}^{*S}(A)))$. This shows that A is a fuzzy β-I-supra openset.

(6) Let A be a fuzzy pre-I-supra openset. Then, we have $A \subseteq \mathrm{Int}^S(\mathrm{Cl}^{*S}(A)) \subseteq \mathrm{Cl}^S(\mathrm{Int}^S(\mathrm{Cl}^{*S}(A)))$. This shows that A is a fuzzy β-I-supra openset. □

Remark 2. The converses of Theorem 18 need not be true as the following examples show.

Example 6. In Example 4, B is a fuzzy α-I-supra openset, but not a fuzzy supra openset.

Example 7. In Example 4, A is a fuzzy pre-I-supra openset, but not a fuzzy-I-supra openset.

Example 8. Let $X = \{a, b, c\}$ and let A, B, and C be a fuzzy sets in X defined as follows:

$$\begin{aligned} A(a) &= 0.7, \\ A(b) &= 0.4, \\ A(c) &= 0.5, \\ B(a) &= 0.2, \\ B(b) &= 0.3, \\ B(c) &= 0.4, \\ C(a) &= 0.8, \\ C(b) &= 0.7, \\ C(c) &= 0.6. \end{aligned} \tag{4}$$

We put $S = \{0_x, 1_x, B\}$. If we take $I = \{0_x\}$, then A is a fuzzy semi-I-supra openset, but not a fuzzy α-I-supra openset.

Example 9. In Example 8, if we put $S = \{0_x, 1_x, A\}$ and we take $I = \{0_x\}$, then C is a fuzzy pre-I-supra openset, but not a fuzzy α-I-supra openset.

Example 10. In Example 3, B is a fuzzy β-I-supra openset, but not a fuzzy semi-I-supra openset.

Example 11. Let $X = \{a, b, c\}$, and let A, B, and C be fuzzy sets in X defined as follows:

$$\begin{aligned} A(a) &= 0.7, \\ A(b) &= 0.4, \\ A(c) &= 0.2, \\ B(a) &= 0.9, \\ B(b) &= 0.8, \\ B(c) &= 0.7, \\ C(a) &= 0.1, \\ C(b) &= 0.2, \\ C(c) &= 0.3. \end{aligned} \tag{5}$$

We put $S = \{0_x, 1_x, A, C, A \cup C\}$. If we take $I = \{0_x\}$, then B is a fuzzy β-I-supra openset, but not a fuzzy pre-I-supra openset.

Theorem 19. *Let (X, S, I) be a fuzzy ideal supra topological space, and let A be any fuzzy set in X. If A is a fuzzy regular-I-supra closedset, then A is a fuzzy semi-I-supra openset.*

Proof. Let A be a fuzzy regular-I-supra closedset. Then, we have $A = (\mathrm{Int}^S(A))^{*S}\backslash\backslash \subseteq (\mathrm{Int}^S(A))^{*S} \cup \mathrm{Int}^S(A) = \mathrm{Cl}^{*S}(\mathrm{Int}^S(A))$. This shows that A is a fuzzy semi-I-supra openset. □

Remark 3. The converses of Theorem 19 need not be true as the following example shows.

Example 12. In Example 11, A is a fuzzy semi-I-supra openset, but not a fuzzy regular-I-supra closedset.

7. Some Fuzzy Ideal Supra Continuous Functions

Definition 23. A function $f\colon (X,S,I) \longrightarrow (Y,\varphi)$ is said to be fuzzy $*$-supra perfectly continuous (resp. fuzzy regular-I-supra closed continuous and fuzzy contra $*$-supra continuous) if for every $V \in \varphi$, $f^{-1}(V)$is a fuzzy $*$-supra perfect (resp. fuzzy regular-I-supra closed and fuzzy S^*-supra closed) set of (X, S, I).

Theorem 20. *For a function $f\colon (X,S,I) \longrightarrow (Y,\varphi)$, the following statements hold:*

(1) Every fuzzy regular-I-supra closed continuous is fuzzy $*$-supra perfectly continuous

(2) Every fuzzy $*$-supra perfectly continuous is fuzzy contra $*$-supra continuous

Proof. This follows from Theorem 17 and Definition 23. □

Remark 4. The converses of Theorem 20 need not be true as shown in the following examples.

Example 13. Let $X = \{a,b,c\}$and$Y = \{x,y,z\}$, and let A and B be fuzzy supra subsets defined as follows:

$$\begin{aligned} A(a) &= 0.7,\\ A(b) &= 0.4,\\ A(c) &= 0.8,\\ B(x) &= 0.3,\\ B(y) &= 0.6,\\ B(z) &= 0.2. \end{aligned} \tag{6}$$

Let $S = \{0_x, 1_x, A\}, \varphi = \{0_Y, 1_Y, B\}$, and $I = \{0_x\}$. Then, the function $f\colon (X,S,I) \longrightarrow (Y,\varphi)$ defined by $f(a) = x$, $f(b) = y$, and $f(c) = z$. Then, f is fuzzy $*$-supra perfectly continuous but not fuzzy regular-I-supra closed continuous.

Example 14. Let $X = \{a,b,c\}$and$Y = \{x,y,z\}$, and let A and B be fuzzy supra subsets defined as follows:

$$\begin{aligned} A(a) &= 0.8,\\ A(b) &= 0.2,\\ A(c) &= 0.4,\\ B(x) &= 0.9,\\ B(y) &= 0.4,\\ B(z) &= 0.7. \end{aligned} \tag{7}$$

Let $S = \{0_x, 1_x, A\}, \varphi = \{0_Y, 1_Y, B\}$, and $I = \{0_x\}$. Then, the function $f\colon (X,S,I) \longrightarrow (Y,\varphi)$ is defined by $f(a) = x$, $f(b) = y$, and $f(c) = z$; then, f is fuzzy contra $*$-supra continuous but not fuzzy $*$-supra perfectly continuous.

Definition 24. A function $f\colon (X,S,I) \longrightarrow (Y,\varphi)$ is said to be fuzzy-I-supra continuous (resp. fuzzy semi-I-supra continuous, fuzzy pre-I-supra continuous, fuzzy α-I-supra continuous, and fuzzy β-I-supra continuous) if for every $V \in \varphi$, $f^{-1}(V)$ is a fuzzy-I-supra openset (resp. fuzzy semi-I-supra open, fuzzy pre-I-supra open, fuzzy α-I-supra open, and fuzzy β-I-supra open) of (X, S, I).

Theorem 21. *For a function $f\colon (X,S,I) \longrightarrow (Y,\varphi)$, the following statements hold:*

(1) Every fuzzy supra continuous is fuzzy α-I-supra continuous

(2) Every fuzzy-I-supra continuous is fuzzy pre-I-supra continuous

(3) Every fuzzy α-I-supra continuous is fuzzy semi-I-supra continuous

(4) Every fuzzy α-I-supra continuous is fuzzy pre-I-supra continuous

(5) Every fuzzy semi-I-supra continuous is fuzzy β-I-supra continuous

(6) Every fuzzy pre-I-supra continuous is fuzzy β-I-supra continuous

Proof. This follows from Theorem 18 and Definition 24. □

Remark 5. The converses of Theorem 21 need not be true as shown in the following examples.

Example 15. In Example 14, f is fuzzy α-I-supra continuous but not fuzzy supra continuous.

Example 16. In Example 14, if we take $I = P(X)$, then f is fuzzy pre-I-supra continuous but not fuzzy-I-supra continuous.

Example 17. Let $X = \{a,b,c\}$and$Y = \{x,y,z\}$, and let A and B be fuzzy supra subsets defined as follows:

$$\begin{aligned} A(a) &= 0.4,\\ A(b) &= 0.1,\\ A(c) &= 0.2,\\ B(x) &= 0.6,\\ B(y) &= 0.9,\\ B(z) &= 0.8. \end{aligned} \tag{8}$$

Let $S = \{0_x, 1_x, A\}, \varphi = \{0_Y, 1_Y, B\}$, and $I = \{0_x\}$. Then, the function $f\colon (X,S,I) \longrightarrow (Y,\varphi)$ is defined by $f(a) = x$, $f(b) = y$, and $f(c) = z$. Then, f is fuzzy semi-I-supra continuous and fuzzy pre-I-supra continuous but not fuzzy α-*I*-supra continuous.

Example 18. In Example 13, f is fuzzy β-I-supra continuous but not fuzzy semi-I-supra continuous.

Example 19. In Example 17, if we take $I = P(X)$, then f is fuzzy β-I-supra continuous but not fuzzy pre-I-supra continuous.

Theorem 22. *Let (X, S, I) be a fuzzy ideal supra topological space and let A be any fuzzy set in X. If A is fuzzy regular-I-supra closedset, then A is a fuzzy semi-I-supra openset.*

Proof. This follows from Theorem 19 and Definitions 23 and 24. □

Remark 6. The converses of Theorem 22 need not be true as the following example shows.

Example 20. In Example 17, A is fuzzy semi-I-supra continuous but not fuzzy regular-I-supra closed continuous.

8. Conclusion

The present paper is focused on the notion of fuzzy ideals in fuzzy supra topological spaces. The concept of a fuzzy s-local function is also introduced here by utilizing the s-neighbourhood structure for a fuzzy supra topological space. These concepts are discussed with a view to find new fuzzy supra topologies from the original one. The basic structure, especially a basis for such generated fuzzy supra topologies, and several relations between different fuzzy ideals and fuzzy supra topologies are also studied here.

Data Availability

The data on fuzzy topological spaces are included within this article.

Conflicts of Interest

The author declares no conflicts of interest.

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