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Research Article

The Measure of Fuzzy Filters on *BL*-Algebras

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The new concept of the fuzzy filter degree was given by means of the implication operator, which enables to measure a degree to which a fuzzy subset of a *BL*-algebra is a fuzzy filter. In this paper, we put forward several equivalent characterizations of the fuzzy filter degree by studying its properties and the relationship with level cut sets. Furthermore, we study the fuzzy filter degrees of the intersection and fuzzy direct products of fuzzy subsets and investigate the fuzzy filter degrees of the image and the preimage of a fuzzy subset under a homomorphism.

1. Introduction

As we know, nonclassical logic algebras are one of the focuses in the basic study of artificial intelligence. Among these logic algebras, *BL*-algebra is one of the most fundamental algebraic structures, which was put forward by Hájek [1] in 1998. *MV*-algebras, Gödel algebras, and product algebras are the most known classes of *BL*-algebras.

Filters theory plays an important role in the theoretical study of these algebras. From logical point of view, a kind of filter corresponds to a set of provable formula. In [2], Liu proposed the notion of fuzzy filters in *BL*-algebras and investigated some of their properties and showed that fuzzy filters are a useful tool to obtain results on classical filters of *BL*-algebras.

In the application of fuzzy theory, a series of concepts of degree were introduced to depict the level of similarity among objects. When these thoughts are applied in the research of fuzzy logic algebras, the question is how to measure the degree to which a fuzzy subset is a fuzzy logic algebra. In 2010, Shi gave an idea in [3]. Shi proposed the concept of fuzzy subgroup degree in the paper to depict the degree to which a fuzzy subset is a fuzzy subgroup. Using this idea, Wang, Shi, Liao, and others have done a lot of work in recent years and got a lot of good results [4–15].

Inspired by the ideas mentioned above, we present a new concept of fuzzy filter, by which the degree to which a fuzzy subset of lattice L is a fuzzy filter is depicted and some academic results are obtained.

This paper is organized as follows: In Section 2, the basic knowledge required in the paper is proposed. In Section 3, we put forward the concept of fuzzy filter degree and its equivalent characterizations. Furthermore, the fuzzy filter degrees of the intersection and fuzzy direct products of fuzzy subsets are investigated and the fuzzy filter degrees of the image and the preimage of a fuzzy subset under a homomorphism are also discussed.

In what follows, let L denote a *BL*-algebra.

2. Preliminaries

In this section, we recollect some basic definitions and results which will be used in the following.

Definition 1 (see [1]). A *BL*-algebra is a structure $(L, \wedge, \vee, \otimes, \longrightarrow, 0, 1)$ such that

(i) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice

(ii) $(L, \otimes, 1)$ is an abelian monoid, i.e., $\otimes$ is commutative and associative and $x \otimes 1 = 1 \otimes x = x$

(iii) The following conditions hold for all $x, y, z \in L$:

(B1) $x \otimes y \leq z$ if and only if $x \leq y \longrightarrow z$ (residuation)

(B2) $x \wedge y = x \otimes (x \longrightarrow y)$ (divisibility)

(B3) $(x \longrightarrow y) \vee (y \longrightarrow x) = 1$ (prelinearity)

Definition 2 (see [1]). Let L be a BL-algebra. A subset F of L is called a filter of L if it satisfies

(1) $1 \in F$

(2) $x \in F$ and $x \longrightarrow y \in F$ imply $y \in F$

Definition 3 (see [1]). Let A be a fuzzy set in L. A is called a fuzzy filter if it satisfies

(1) $A(1) \geq A(x)$ for all $x \in L$

(2) $A(y) \geq A(x) \wedge A(x \longrightarrow y)$ for all $x, y \in L$

Definition 4 (see [1]). *A mapping f: $L \longrightarrow M$ from a BL-algebra L* into a BL-algebra M is called a BL homomorphism if for any $x, y \in L$:

(1) $f(0_L) = 0_M$

(2) $f(x \otimes y) = f(x) \otimes f(y)$

(3) $f(x \longrightarrow y) = f(x) \longrightarrow f(y)$

where 0_L and 0_M are the least elements of L and M, respectively.

If f is a surjection, then f is called an epimorphism. If f is an injective, then f is called a monomorphism. If f is a bijective, then f is called an isomorphism.

It is easy to verify that $f(1_L) = 1_M$, where 1_L and 1_M are the largest elements of L and M, respectively.

Definition 5 (see [16]). A mapping I: $[0,1] \times [0,1] \longrightarrow [0,1]$ is called a fuzzy implication operator, if it satisfies

(1) $x \leq y$ implies $I(x,z) \geq I(y,z)$ for all $x, y, z \in [0,1]$

(2) $y \leq z$ implies $I(x,y) \leq I(y,z)$ for all $x, y, z \in [0,1]$

(3) $I(1,0) = 0$, $I(0,0) = I(1,1) = 1$

For any $x, y \in [0,1]$, let $I(x,y) = \vee \{t \in [0,1] | x \wedge t \leq y\}$. Obviously, I is an implication, which is called the R-implication generated by "min," and is usually denoted as $x \longrightarrow y$.

The implication operator has the following properties.

Proposition 1 (see [17]). *For any a, $a_i (i \in I) \in [0,1]$, the following hold:*

(1) $a \longrightarrow \bigwedge_{i\in I} a_i = \bigwedge_{i\in I} (a \longrightarrow a_i)$

(2) $\bigvee_{i\in I} a_i \longrightarrow a = \bigwedge_{i\in I} (a_i \longrightarrow a)$

(3) $(\bigwedge_{i\in I} a_i) \wedge (\bigwedge_{j\in I} b_j) = \bigwedge_{i,j\in I} (a_i \wedge b_j)$

Definition 6 (see [18]). Let A be a fuzzy set in L, $b \in [0,1]$, then the set $A_{[b]} = \{x \in L | A(x) \geq b\}$ is called a level subset of A; the set $A_{(b)} = \{x \in L \,|\, A(x) > b\}$ is called a strong level subset of A.

Definition 7 (see [16]) (extension principle). With the mapping $f : X \longrightarrow Y$, two mappings can be induced by this mapping, respectively, denoted as f and f^{-1} as follows:

$$f: F(X) \longrightarrow F(Y), A \longmapsto f(A), \qquad f^{-1}: F(Y) \longrightarrow F(X), B \longmapsto f^{-1}(B), \tag{1}$$

where the membership function of $f(A)$ and $f^{-1}(B)$ is defined as

$$f(A)(y) = \bigvee_{f(x)=y} A(x), \qquad f^{-1}(B)(x) = B(f(x)). \tag{2}$$

Definition 8 (see [19]). Let $L_i (i \in I)$ be a family of BL-algebras. $\prod_{i\in I} L_i$ is defined as $\{x \,|\, x : I \longrightarrow \cup_{i\in L} L_i, x_i \triangleq x(i) \in L_i, i \in I\}$. The operations "$\vee$, $\wedge$, $\otimes$, and $\longrightarrow$" are given as follows:

$$\begin{aligned} (x \vee y)(i) &= x(i) \vee y(i) = x_i \vee y_i, \\ (x \wedge y)(i) &= x(i) \wedge y(i) = x_i \wedge y_i, \\ (x \otimes y)(i) &= x(i) \otimes y(i) = x(i) \otimes y(i), \\ (x \longrightarrow y)(i) &= x(i) \longrightarrow y(i) = x_i \longrightarrow y_i, \\ O(i) &= 0_i, \\ I(i) &= 1_i, \end{aligned} \tag{3}$$

where 1_i and 0_i are the largest element and the least element of L_i, respectively. $(\prod_{i\in I} L_i, \vee, \wedge)$ is called the BL direct product.

It is easy to verify that $(\prod_{i\in I} L_i, \vee, \wedge, \otimes, \longrightarrow, O, I)$ is a BL-algebra.

Definition 9 (see [20]). Let A_i be a fuzzy subset of L_i $(i \in I)$. The fuzzy subset $\prod_{i\in I} A_i$ of $\prod_{i\in I} L_i$ is defined as

$$\forall x = \prod_{i\in I} x_i \in \prod_{i\in I} L_i (x_i \in L_i), \left(\prod_{i\in I} A_i\right)(x) = \bigwedge_{i\in I} A_i(x_i), \tag{4}$$

where $\prod_{i\in I} A_i$ is called the fuzzy direct product of A_i $(i \in I)$.

Definition 10 (see [21]). Let f be a mapping of a BL-algebra L into a BL-algebra M. Then, a fuzzy subset A of L is called f-invariant if for all $x, y \in L$, and $f(x) = f(y)$ implies $A(x) = A(y)$.

3. Fuzzy Filters Degree

Definition 11. Let A be a fuzzy subset in L. The fuzzy filter degree $m_L(A)$ of A is defined as

$$m_L(A) = \left[\bigwedge_{x,y\in L}\{A(x)\wedge A(x\longrightarrow y)\longrightarrow A(y)\}\right] \wedge \left[\bigwedge_{x\in L}\{A(x)\longrightarrow A(1)\}\right] = \left[\bigwedge_{x,y\in L}\vee\{t\in[0,1] \mid A(x)\wedge A(x\longrightarrow y)\wedge t\le A(y)\}\right] \wedge \left[\bigwedge_{x\in L}\vee\{t\in[0,1]\mid A(x)\wedge t\le A(1)\}\right]. \quad (5)$$

For convenience, we denote

$$B_1(x,y) = \{t\in[0,1]\mid A(x)\wedge A(x\longrightarrow y)\wedge t\le A(y)\},$$
$$B_2(x) = \{t\in[0,1]\mid A(x)\wedge t\le A(1)\}. \quad (6)$$

So, $m_L(A) = [\bigwedge_{x,y\in L}\vee B_1(x,y)]\wedge[\bigwedge_{x\in L}\vee B_2(x)]$.

Remark 1. In Definition 11, $m_L(A)$ can be used to measure the degree of a fuzzy subset A to be a fuzzy filter of L.

Example 1. Let $L=\{0,a,b,1\}$ be a lattice, and the order "$\le$" on L be determined by using Figure 1. For all $x,y\in L$, the two binary operations $\longrightarrow$ and $\otimes$ are defined by Table 1 and 2. Then, $(L,\vee,\wedge,\otimes,\longrightarrow,0,1)$ is a BL-algebra.

Let A_1, A_2, and A_3 be the fuzzy sets in L given by

$$A_1 = \left\{\frac{0.2}{0},\frac{0.2}{a},\frac{0.6}{b},\frac{0.7}{1}\right\},$$
$$A_2 = \left\{\frac{0.2}{0},\frac{0.4}{a},\frac{0.4}{b},\frac{0.7}{1}\right\}, \quad (7)$$
$$A_3 = \left\{\frac{0.3}{0},\frac{0.4}{a},\frac{0.4}{b},\frac{0.7}{1}\right\}.$$

One can easily check that A_1 is a fuzzy filter of L. Neither A_2 nor A_3 is a fuzzy filter of L. From Definition 11, we have $m_L(A_1)=1$, $m_L(A_2)=0.2$, and $m_L(A_3)=0.3$.

Since $I(x,y)$ is a continuous t-norm on [0,1], it is not difficult to have the following.

Theorem 1. *Let A be a fuzzy subset of L. A is a fuzzy filter of L if and only if* $m_L(A)=1$.

The properties of $m_L(A)$ are discussed below, and their equivalent characterizations are also given.

Lemma 1. *Let A be a fuzzy subset of L.* $m_L(A)\ge a$ *if and only if for all* $x,y\in L$, $A(x)\wedge A(x\longrightarrow y)\wedge a\le A(y)$ *and* $A(x)\wedge a\le A(1)$.

Proof. Suppose $m_L(A)\ge a$ and then $[\bigwedge_{x,y\in L}\vee B_1(x,y)]\wedge[\bigwedge_{x\in L}\vee B_2(x)]\ge a$ by Definition 11. Thus, $\bigwedge_{x,y\in L}\vee B_1(x,y)\ge a$ and $\bigwedge_{x\in L}\vee B_2(x)\ge a$.

Note $b_1(x,y) = \vee B_1(x,y)$. There exists $t_\varepsilon\in B_1(x,y)$ for any $\varepsilon>0$ such that $t_\varepsilon > b_1(x,y)-\varepsilon$. Therefore, since $t_\varepsilon\in B_1(x,y)$, we have $A(x)\wedge A(x\longrightarrow y)\wedge t_\varepsilon\le A(y)$. So, $A(x)\wedge A(x\longrightarrow y)\wedge(b_1(x,y)-\varepsilon)\le A(x)\wedge A(y)\wedge t_\varepsilon\le A(y)$.

By the arbitrariness of ε, $A(x)\wedge A(x\longrightarrow y)\wedge b_1(x,y)\le A(y)$ as well as $b_1(x,y)\ge a$; therefore, $A(x)\wedge A(x\longrightarrow y)\wedge a\le A(y)$. Similarly, we can prove $A(x)\wedge a\le A(1)$.

Conversely, if $A(x)\wedge A(x\longrightarrow y)\wedge a\le A(y)$ and $A(x)\wedge a\le A(1)$ for any x, $y\in L$, then $a\in B_1(x,y)$ and $a\in B_2(x)$. Hence, we have

$$m_L(A)\left[\bigwedge_{x,y\in L}\vee B_1(x,y)\right]\wedge\left[\bigwedge_{x\in L}\vee B_2(x)\right]\ge a. \quad (8)$$

The proof is completed. □

Theorem 2. *Suppose A be a fuzzy subset of L and*

$$K_1 = \{a\in(0,1)]\mid A(x)\wedge A(x\longrightarrow y)\wedge a\le A(y),\quad \forall x,y\in L\},$$
$$K_2 = \{a\in(0,1)]\mid A(x)\wedge a\le A(1),\quad \forall x\in L\},$$
$$K = \{a\in(0,1)]\mid A(x)\wedge A(x\longrightarrow y)\wedge a\le A(y), A(x)\wedge a\le A(1), \forall x,y\in L\}. \quad (9)$$

Then, $m_L(A) = (\vee K_1)\wedge(\vee K_2) = \vee K$.

Proof. We first show that $m_L(A) = \vee K$.

Let $m_L(A)=c$. By Lemma 1, we get that for any $x,y\in L$, $A(x)\wedge A(x\longrightarrow y)\wedge c\le A(y)$ and $A(x)\wedge c\le A(1)$.

Let $b=\vee K$, and it is easy to get $b\ge c$. There exists $t_\varepsilon\in K$ for any $\varepsilon>0$ such that $t_\varepsilon>b-\varepsilon$. Since $t_\varepsilon\in K$, we have $A(x)\wedge A(x\longrightarrow y)\wedge t_\varepsilon\le A(y)$ and $A(x)\wedge t_\varepsilon\le A(1)$ for any x, $y\in L$. So, $A(x)\wedge A(x\longrightarrow y)\wedge(b-\varepsilon)\le A(y)$ and $A(x)\wedge(b-\varepsilon)\le A(1)$.

By the arbitrariness of ε, we have

$$A(x)\wedge A(x\longrightarrow y)\wedge b\le A(y),$$
$$A(x)\wedge b\le A(1). \quad (10)$$

Moreover, $m_L(A)\ge b$ and $c\ge b$ by Lemma 1. So, $b=c$.

Now we prove $(\vee K_1)\wedge(\vee K_2) = \vee K$.

As $A(x)\wedge A(x\longrightarrow y)\wedge a\le A(y)$ and $A(x)\wedge a\le A(1)$ for any $a\in K$ and $x,y\in L$, we have $a\in K_1$ and $a\in K_2$. Then, $a\le\vee K_1$ and $a\le\vee K_2$. Thus, $a\le(\vee K_1)\wedge(\vee K_2)$. Therefore, $\vee K\le(\vee K_1)\wedge(\vee K_2)$.

Let $a_i = \vee K_1 (i=1,2)$. There exists $t_i\in K_i (i=1,2)$, such that $t_i>a_i-\varepsilon$ for any $\varepsilon>0$.

Let $b=(\vee K_1)\wedge(\vee K_2)$. Then, $a_i\ge b (i=1,2)$. So, $t_i>b-\varepsilon$.

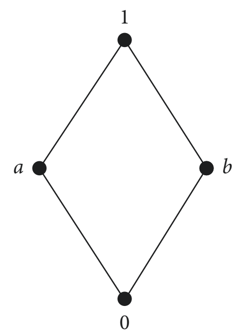

Figure 1: The order "≤" on L.

Table 1: The binary operation "⊗".

⊗	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Table 2: The binary operation "⟶".

⟶	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Thus, $A(x) \wedge A(x \longrightarrow y) \wedge (b-\varepsilon)a \leq A(y)$ and $A(x) \wedge (b-\varepsilon) \leq A(1)$.

By the arbitrariness of ε, we have

$$\begin{aligned} A(x) \wedge A(x \longrightarrow y) \wedge b &\leq A(y), \\ A(x) \wedge b &\leq A(1). \end{aligned} \tag{11}$$

By Lemma 1, we get that $m_L(A) \geq b$, that is, $\vee K \geq b$

Finally, we have $(\vee K_1) \wedge (\vee K_2) = \vee K$.

We know there is a close relationship between the fuzzy filter of a BL-algebra and its level set. Next, we give some characterizations of the fuzzy filter degree of a fuzzy set by means of its level sets. □

Lemma 2. *Let A be a fuzzy subset of L. If $m_L(A) = c > 0$, then nonvoid $A_{[b]}$ is a filter of L for any $b \in (0, c]$.*

Proof. If $A_{[b]} \neq \varnothing$ for any $b \in (0, c]$; then, for any x, $x \longrightarrow y \in A_{[b]}$, we have $A(x) \geq b$ and $A(x \longrightarrow y) \geq b$. Since $m_L(A) = c$, the followings are obtained by Lemma 1:

$$\begin{aligned} A(y) &\geq A(x) \wedge A(x \longrightarrow y) \wedge c \geq b \wedge c = b, \\ A(1) &\geq A(x) \wedge c \geq b \wedge c = b. \end{aligned} \tag{12}$$

So, $y \in A_{[b]}$ and $1 \in A_{[b]}$. Hence, $A_{[b]}$ is a filter of L.

For strong level sets, we can get the similar conclusions such as Lemma 3 by suitable modification to the proof of Lemma 2. □

Lemma 3. *Let A be a fuzzy subset of L. If $m_L(A) = c > 0$, then nonvoid $A_{(b)}$ is a filter of L for any $b \in (0, c]$.*

Theorem 3. *Let A be a fuzzy subset of L. If $m_L(A) = c > 0$, then*
$m_L(A) = \vee \{ a \in (0,1] \mid \forall b \in (0,a],\ A_{[b]} \neq \varnothing \text{ is a filter of } L \}$.

Proof. Suppose that $m_L(A) = c$ and $B = \{a \in (0,1] \mid \forall b \in (0,a],\ A_{[b]} \neq \varnothing$ is a filter of $L\}$. According to Lemma 2, we have $c \in B$; thus, $\vee B \geq c$, i.e., $\vee B \geq m_L(A)$

Next, we prove $\vee B \leq m_L(A)$.

In fact, for any $a \in B$ and x, $y \in L$, we have $A(x) \wedge A(x \longrightarrow y) \wedge a \leq A(y)$. Otherwise, there exists x_0, $y_0 \in L$, such that $A(x_0) \wedge A(x_0 \longrightarrow y_0) \wedge a > A(y_0)$.

Let $A(x_0) \wedge A(x_0 \longrightarrow y_0) \wedge a = b$. Then, $b \in (0, a]$, $A(x_0) \wedge A(x_0 \longrightarrow y_0) \geq b$, and $A(y_0) < b$. That is, x_0, $x_0 \longrightarrow y_0 \in A_{[b]}$ and $A(y_0) < b$.

Since $a \in B$ and, $b \in (0, a]$ $A_{[b]}$ is a filter of L. This together with x_0, $x_0 \longrightarrow y_0 \in A_{[b]}$ implies that $y_0 \in A_{[b]}$, i.e., $A(y_0) \geq b$. This is contrary to $A(y_0) < b$

Similarly, for any $a \in B$, $x \in L$, we have $A(x) \wedge a \leq A(1)$. By Lemma 1, it follows that $m_L(A) \geq a$. Thus, $\vee B \leq m_L(A)$.

From the above, we have $m_L(A) = \vee B = \vee \{ a \in (0,1] \mid \forall b \in (0,a], A_{[b]} \neq \varnothing \text{ is a filter of } L \}$.

With Lemma 1 and Lemma 3, proceeding similar as in the proof of Theorem 3, we have the following theorem. □

Theorem 4. *Let A be a fuzzy subset of L. Then, $m_L(A) = \vee \{ a \in (0,1] \mid \forall b \in (0,a], A_{[b]} \neq \varnothing \text{ is a filter of } L \}$.*

Next, we discuss the fuzzy filter degree under fuzzy subset operations.

Firstly, the relation of the fuzzy filter degree between the intersection of a family of fuzzy subsets and each fuzzy subset is given.

Theorem 5. *Let $\{A_i\}_{i\in I}$ be a family of fuzzy subsets of L. Then,*

$$m_L\Big(\bigcap_{i\in I} A_i\Big) \geq \bigwedge_{i\in I} m_L(A_i). \tag{13}$$

Proof. By Proposition 1 and Definition 5, we have

$$\begin{aligned}
&\bigwedge_{x,y\in L} \Big\{\Big(\bigcap_{i\in I} A_i\Big)(x) \wedge \Big(\bigcap_{i\in I} A_i\Big)(x \longrightarrow y) \longrightarrow \Big(\bigcap_{i\in I} A_i\Big)(y)\Big\} \\
&= \bigwedge_{x,y\in L} \Big\{\Big(\bigwedge_{i\in I} A_i(x)\Big) \wedge \Big(\bigwedge_{i\in I} A_i(x \longrightarrow y)\Big) \longrightarrow \bigwedge_{i\in I} A_i(y)\Big\} \\
&= \bigwedge_{x,y\in L} \bigwedge_{i\in I} \Big\{\Big(\bigwedge_{i\in I} A_i(x)\Big) \wedge \Big(\bigwedge_{i\in I} A_i(x \longrightarrow y)\Big) \longrightarrow A_i(y)\Big\} \text{ (by Proposition 1)} \\
&\geq \bigwedge_{x,y\in L} \bigwedge_{i\in I} \{A_i(x) \wedge A_i(x \longrightarrow y) \longrightarrow A_i(y)\} \text{ (by Definition 5)} \\
&= \bigwedge_{i\in I} \bigwedge_{x,y\in L} \{A_i(x) \wedge A_i(x \longrightarrow y) \longrightarrow A_i(y)\} \\
&\geq \bigwedge_{i\in I} \Big\{\Big[\bigwedge_{x,y\in L} \{A_i(x) \wedge A_i(x \longrightarrow y) \longrightarrow A_i(y)\}\Big] \wedge \Big[\bigwedge_{i\in L} \{A_i(x) \longrightarrow A_i(1)\}\Big]\Big\} \\
&= \bigwedge_{i\in I} m_L(A_i).
\end{aligned} \tag{14}$$

Following in a similar manner, we can get

$$\bigwedge_{x,y\in L} \Big\{\Big(\bigcap_{i\in I} A_i\Big)(x) \longrightarrow \Big(\bigcap_{i\in I} A_i\Big)(1)\Big\} \geq \bigwedge_{i\in I} m_L(A_i). \tag{15}$$

From Definition 11, it follows that

$$\begin{aligned}
m_L\Big(\bigcap_{i\in I} A_i\Big) &= \Big[\bigwedge_{x,y\in L} \Big\{\Big(\bigcap_{i\in I} A_i\Big)(x) \wedge \Big(\bigcap_{i\in I} A_i\Big)(x \longrightarrow y) \\
&\longrightarrow \Big(\bigcap_{i\in I} A_i\Big)(y)\Big\}\Big] \wedge \Big[\bigwedge_{x,y\in L} \Big\{\Big(\bigcap_{i\in I} A_i\Big)(x) \\
&\longrightarrow \Big(\bigcap_{i\in I} A_i\Big)(1)\Big\}\Big] \geq \bigwedge_{i\in I} m_L(A_i).
\end{aligned} \tag{16}$$

Secondly, the fuzzy filter degree of fuzzy subsets under the fuzzy direct product is discussed. □

Theorem 6. *Let A_i be a fuzzy subset of $L_i (i \in I)$; then, $m_L(\prod_{i\in I} A_i) \geq \bigwedge_{i\in I} m_{L_i}(A_i)$.*

Proof. Suppose $x, y \in \prod_{i\in I} L_i = L$. Then, $x = \prod_{i\in I} x_i$ and $y = \prod_{i\in I} y_i$, where $x_i, y_i \in L_i$.

$$\begin{aligned}
&\bigwedge_{x,y\in L} \Big\{\Big(\prod_{i\in I} A_i\Big)(x) \wedge \Big(\prod_{i\in I} A_i\Big)(x \longrightarrow y) \longrightarrow \Big(\prod_{i\in I} A_i\Big)(y)\Big\} \\
&= \bigwedge_{x_i,y_i\in L_i} \Big\{\Big(\bigwedge_{i\in I} A_i(x_i)\Big) \wedge \Big(\bigwedge_{i\in I} A_i(x_i \longrightarrow y_i)\Big) \longrightarrow \bigwedge_{i\in I} A_i(y_i)\Big\} \\
&= \bigwedge_{x_i,y_i\in L_i} \bigwedge_{i\in I} \Big\{\Big(\bigwedge_{i\in I} A_i(x_i)\Big) \wedge \Big(\bigwedge_{i\in I} A_i(x_i \longrightarrow y_i)\Big) \longrightarrow A_i(y_i)\Big\} \quad \text{(by Proposition 1)} \\
&\geq \bigwedge_{x_i,y_i\in L_i} \bigwedge_{i\in I} \{A_i(x_i) \wedge A_i(x_i \longrightarrow y_i) \longrightarrow A_i(y_i)\} \quad \text{(by Definition 5)} \\
&= \bigwedge_{i\in I} \bigwedge_{x_i,y_i\in L_i} \{A_i(x_i) \wedge A_i(x_i \longrightarrow y_i) \longrightarrow A_i(y_i)\} \\
&\geq \bigwedge_{i\in I} \Big\{\Big[\bigwedge_{x_i,y_i\in L_i} \{A_i(x_i) \wedge A_i(x_i \longrightarrow y_i) \longrightarrow A_i(y_i)\}\Big] \wedge \Big[\bigwedge_{x_i\in L_i} \{A_i(x_i) \longrightarrow A_i(1)\}\Big]\Big\} \\
&= \bigwedge_{i\in I} m_{L_i}(A_i).
\end{aligned} \tag{17}$$

Following in a similar manner, we can get

$$\bigwedge_{x,y\in L}\left\{\left(\prod_{i\in I}A_i\right)(x)\longrightarrow\left(\prod_{i\in I}A_i\right)(1)\right\}\geq\bigwedge_{i\in I}m_{L_i}(A_i). \tag{18}$$

So, by Definition 11,

$$\begin{aligned}m_L\left(\prod_{i\in I}A_i\right)&=\bigwedge_{x,y\in L}\left\{\left(\prod_{i\in I}A_i\right)(x)\wedge\left(\prod_{i\in I}A_i\right)(x\longrightarrow y)\longrightarrow\left(\prod_{i\in I}A_i\right)(y)\right\}\\&\wedge\left[\bigwedge_{x,y\in L}\left\{\left(\prod_{i\in I}A_i\right)(x)\longrightarrow\left(\prod_{i\in I}A_i\right)(1)\right\}\right]\\&\geq\bigwedge_{i\in I}m_{L_i}(A_i).\end{aligned} \tag{19}$$

Finally, we give the relation between the fuzzy ideal degrees of the image and the preimage of a fuzzy subset under a homomorphism. □

Theorem 7. *Let f: $L_1\longrightarrow L_2$ be a BL homomorphism, and A and B be the fuzzy subsets of L_1 and L_2, respectively. The following assertions hold:*

(1) If f is a surjective and f-invariant, then $m_{L_1}(A)\leq m_{L_2}(f(A))$
(2) $m_{L_2}(B)\leq m_{L_1}(f^{-1}(B))$
(3) If f is a surjective, then $m_{L_2}(B)\leq m_{L_1}(f^{-1}(B))$

Proof. (1) By Theorem 2 and Definition 7, we can have that

$$\begin{aligned}m_{L_2}(f(A))&=\vee\left\{a\in[0,1]\,\middle|\,f(A)(x')\wedge f(A)(x'\longrightarrow y')\wedge a\leq f(A)(y'),f(A)(x')\wedge a\leq f(A)(1),\quad\forall x',y'\in L_2\right\}\\&=\vee\left\{a\in(0,1]\,\middle|\,\left(\bigvee_{f(x)=x'}A(x)\right)\wedge\left(\bigvee_{f(x)=x'\longrightarrow y'}A(x)\right)\wedge a\leq\bigvee_{f(x)=y'}A(x),\ \bigvee_{f(x)=x'}A(x)\wedge a\leq\bigvee_{f(x)=1}A(x),\quad\forall x',y'\in L_2\right\}.\end{aligned} \tag{20}$$

Since f is an epimorphism, there exists x_0, $y_0\in L_1$ such that $f(x_0)=x'$ and $f(y_0)y'$.
Thus, $f(x_0\longrightarrow y_0)=f(x_0)\longrightarrow f(y_0)=x'\longrightarrow y'$. Now that f is an f-invariant, which together with $f(1)=1$, we have

$$\begin{aligned}&\bigvee_{f(x)=x'}A(x)=A(x_0),\\&\bigvee_{f(x)=y'}A(x)=A(y_0),\\&\bigvee_{f(x)=x'\longrightarrow y'}A(x)=A(x_0\longrightarrow y_0),\\&\bigvee_{f(x)=1}A(x)=A(1).\end{aligned} \tag{21}$$

It follows that

$$\begin{aligned}m_{L_2}(f(A))&=\vee\left\{a\in(0,1]\,\middle|\,A(x_0)\wedge A(x_0\longrightarrow y_0)\wedge a\right.\\&\left.\leq A(y_0),\quad A(x_0)\wedge a\leq A(1)\right\}\\&\geq\vee\left\{a\in[0,1]\,\middle|\,[A(x)\wedge A(x\longrightarrow y)]\wedge a\right.\\&\left.\leq A(y),\quad A(x)\wedge a\leq A(1),\forall x,y\in L_1\right\}\\&=m_{L_1}(A).\end{aligned} \tag{22}$$

(2) By Definition 11 and Definition 7, we can have that

$$
\begin{aligned}
m_{L_1}\left(f^{-1}(B)\right) &= \left[\bigwedge_{x,y\in L_1}\left\{f^{-1}(B)(x)\wedge f^{-1}(B)(x\longrightarrow y)\longrightarrow f^{-1}(B)(y)\right\}\right] \\
&\quad \wedge\left[\bigwedge_{x\in L_1}\left\{f^{-1}(B)(x)\longrightarrow f^{-1}(B)(1)\right\}\right] \\
&= \left[\bigwedge_{x,y\in L_1}\{B(f(x))\wedge B(f(x\longrightarrow y))\longrightarrow B(f(y))\}\right]\wedge\left[\bigwedge_{x\in L_1}\{B(f(x))\longrightarrow B(f(1))\}\right] \\
&= \left[\bigwedge_{x,y\in L_1}\{B(f(x))\wedge B(f(x)\longrightarrow f(y))\longrightarrow B(f(y))\}\right] \\
&\quad \wedge\left[\bigwedge_{x\in L_1}\{B(f(x))\longrightarrow B(1)\}\right] \quad \text{(by Definition 4)} \\
&\geq \left[\bigwedge_{x',y'\in L_2}\{B(x')\wedge B(x'\longrightarrow y')\longrightarrow B(y')\}\right]\wedge\left[\bigwedge_{x'\in L_2}\{B(x')\longrightarrow B(1)\}\right] = m_{L_2}(B).
\end{aligned}
\tag{23}
$$

(3) If f is an epimorphism, then from the proof of (2), we have

$$
\begin{aligned}
m_{L_1}\left(f^{-1}(B)\right) &= \left[\bigwedge_{x,y\in L_1}\{B(f(x))\wedge B(f(x)\longrightarrow f(y))\longrightarrow B(f(y))\}\right]\wedge\left[\bigwedge_{x\in L_1}\{B(f(x))\longrightarrow B(1)\}\right] \\
&= \left[\bigwedge_{x',y'\in L_2}\{B(x')\wedge B(x'\longrightarrow y')\longrightarrow B(y')\}\right]\wedge\left[\bigwedge_{x'\in L_2}\{B(x')\longrightarrow B(1)\}\right] = m_{L_2}(B).
\end{aligned}
\tag{24}
$$

□

4. Conclusions

In order to measure the degree to which a fuzzy subset is a fuzzy filter, we put forward the concept of the fuzzy filter degree of L. Through the research on the fuzzy filter degree of L, we have a deeper understanding about the fuzzy filter, further enriching the theory of fuzzy filter of BL-algebras. Using this idea, we can also measure other fuzzy algebraic structures and it also provides theoretical basis for fuzzy pattern recognition and other applications.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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