Hindawi
Advances in Fuzzy Systems
Volume 2019, Article ID 5925036, 7 pages
https://doi.org/10.1155/2019/5925036

Research Article

L-Fuzzy Prime Ideals in Universal Algebras

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Received 30 December 2018; Accepted 10 March 2019; Published 1 April 2019

Academic Editor: Antonin Dvorák

In this paper, the notions of $L-$fuzzy prime ideals and maximal $L-$fuzzy ideals of universal algebras are introduced by applying the general theory of algebraic fuzzy systems.

1. Introduction

The commutator (or the product) of ideals I and J of a ring R, written as IJ, is the ideal of R generated by all products ij and ji, with $i \in I$ and $j \in J$; i.e.,

$$IJ = \{x \in R : x = \Sigma_{i=1}^{n} y_i z_i, y_i \in I, z_i \in J\}. \quad (1)$$

In 1984, H. P. Gumm and A. Ursini [1] have studied the commutator (or the product) of ideals in a more general context. They have defined and characterized the commutator of ideals in universal algebras by the use of commutator terms. Later on, A. Ursini [2] applied this product to study prime ideals of universal algebras. P. Agliano [3] then studied the prime spectrum of universal algebras.

The concept of fuzzy sets was first introduced by Zadeh [4] and this concept was adapted by Rosenfeld [5] to define fuzzy subgroups. Since then, many authors have been studying fuzzy subalgebras of several algebraic structures (see [6–9]). As suggested by Gougen [10], the unit interval $[0, 1]$ is not sufficient to take the truth values of general fuzzy statements. U. M. Swamy and D. V. Raju [11, 12] studied the general theory of algebraic fuzzy systems by introducing the notion of a fuzzy $\mathfrak{L}-$ subset of a set X corresponding to a given class $\mathfrak{L}$ of subsets of X having truth values in a complete lattice satisfying the infinite meet distributive law. Swamy and Swamy [13] defined the commutator (or the product) of $L-$fuzzy ideals μ and σ of a ring R as follows:

$$[\mu, \sigma](x) = \bigvee \left\{ \bigwedge_{i=1}^{n} (\mu(y_i) \wedge \sigma(z_i)) : x = \Sigma_{i=1}^{n} y_i z_i \right\} \quad (2)$$

for all $x \in R$. They have used this commutator to define $L-$fuzzy prime ideals of rings.

In [14], we have studied $L-$fuzzy ideals in universal algebras having a definable constant denoted by 0, where L is a complete distributive lattice satisfying the infinite meet distributive law. We gave a necessary and sufficient condition for a class of algebras to be ideal-determined. In the present paper, we define the commutator of $L-$fuzzy ideals in universal algebras and investigate some of its properties. Moreover, we study $L-$fuzzy prime ideals and maximal $L-$fuzzy ideals in universal algebras as a generalization of $L-$fuzzy prime ideals in those well-known structures: in semigroups [15], in rings [13], in semirings [16], in ternary semirings [17], in Γ-rings [18], in modules [19], in lattices [9], and in other algebraic structures.

2. Preliminaries

This section contains some definitions and results which will be used in this paper. For those elementary concepts on universal algebras we refer to [20, 21]. Throughout this paper $A \in \mathscr{K}$, where $\mathscr{K}$ is a class of algebras of a fixed type Ω, and we assume that there is an equationally definable constant in all algebras of $\mathscr{K}$ denoted by 0. For a positive integer n, we write $\overrightarrow{a}$ to denote the $n-$tuple $\langle a_1, a_2, \ldots, a_n\rangle \in A^n$.

Definition 1 ([1]). A term $P(\overrightarrow{x}, \overrightarrow{y})$ is said to be an ideal term in $\overrightarrow{y}$ if and only if $P(\overrightarrow{x}, \overrightarrow{0}) = 0$.

Definition 2 ([1]). A nonempty subset I of A is called an ideal of A if and only if $P(\overrightarrow{a}, \overrightarrow{b}) \in I$ for all $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in I^m$ and any ideal term $P(\overrightarrow{x}, \overrightarrow{y})$ in $\overrightarrow{y}$.

We denote the class of all ideals of A, by $\mathscr{I}(A)$.

Definition 3 ([1]). A class $\mathscr{K}$ of algebras is called ideal-determined if every ideal I is the zero congruence class of a unique congruence relation denoted by I^δ. In this case the map $I \longmapsto I^\delta$ defines an isomorphism between the lattice of ideals and congruences on A.

Definition 4 ([1, 2]). A term $t(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z})$ is said to be a commutator term in $\overrightarrow{y}, \overrightarrow{z}$ if and only if it is an ideal term in $\overrightarrow{y}$ and an ideal term in $\overrightarrow{z}$.

Definition 5 ([1]). In an ideal-determined variety, the commutator $[I, J]$ of ideals I and J is the zero congruence class of the commutator congruence $[I^\delta, J^\delta]$.

It is characterized in [1] as follows.

Theorem 6 ([1, 2]). *In an ideal-determined variety,*

$$[I, J] = \left\{ t\left(\overrightarrow{a}, \overrightarrow{i}, \overrightarrow{j}\right) : \overrightarrow{a} \in A^n, \overrightarrow{i} \in I^m \text{ and } \overrightarrow{j} \in J^k \text{ where } t\left(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}\right) \text{ is a commutator term in } \overrightarrow{y}, \overrightarrow{z} \right\}. \tag{3}$$

For subsets H, G of A, $[H, G]$ denotes the product $[\langle H\rangle, \langle G\rangle]$. In particular, for $a, b \in A$, $[\langle a\rangle, \langle b\rangle]$ is denoted by $[a, b]$.

Definition 7 ([2]). A proper ideal P of A is called prime if and only if for all $I, J \in \mathscr{I}(A)$

$$[I, J] \subseteq P \Longrightarrow \text{ either } I \subseteq P \text{ or } J \subseteq P. \tag{4}$$

Theorem 8 ([2]). *A proper ideal P of A is prime if and only if*

$$[a, b] \subseteq P \Longrightarrow \text{ either } a \in P \text{ or } b \in \mathrm{P} \tag{5}$$

for all $a, b \in A$.

Throughout this paper $L = (L, \wedge, \vee, 0, 1)$ is a complete Brouwerian lattice; i.e., L is a complete lattice satisfying the infinite meet distributive law. By an L–fuzzy subset of A, we mean a mapping $\mu : A \longrightarrow L$. For each $\alpha \in L$, the α–level set of μ denoted by μ_α is a subset of A given by the following.

$$\mu_\alpha = \{x \in A : \alpha \le \mu(x)\} \tag{6}$$

For L–fuzzy subsets μ and ν of A, we write $\mu \le \nu$ to mean $\mu(x) \le \nu(x)$ in the ordering of L.

Definition 9 ([22]). For each $x \in A$ and $0 \ne \alpha$ in L, the L–fuzzy subset x_α of A given by

$$x_\alpha(z) = \begin{cases} \alpha & \text{if } z = x \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

is called the L–fuzzy point of A. In this case x is called the support of x_α and α is its value.

For an L–fuzzy subset μ of A and an L–fuzzy point x_α of A, we write $x_\alpha \in \mu$ whenever $\mu(x) \ge \alpha$.

Definition 10 ([14]). An L–fuzzy subset μ of A is said to be an L–fuzzy ideal of A if and only if the following conditions are satisfied:

(1) $\mu(0) = 1$, and

(2) If $P(\overrightarrow{x}, \overrightarrow{y})$ is an ideal term in $\overrightarrow{y}$ and $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in A^m$; then

$$\mu\left(P\left(\overrightarrow{a}, \overrightarrow{b}\right)\right) \ge \mu^m\left(\overrightarrow{b}\right). \tag{8}$$

Note that an L–fuzzy subset μ of A satisfying the two conditions in the above definition can be regarded as a normal L–fuzzy ideal in the sense of Jun et al. [23].

Theorem 11 ([14]). *A class $\mathscr{K}$ of algebras is ideal-determined if and only if every L–fuzzy ideal μ is the zero fuzzy congruence class of a unique L–fuzzy congruence relation denoted by Θ^μ.*

3. The Commutator of L–Fuzzy Ideals

In this section, we define the commutator (or the product) of L–fuzzy ideals in universal algebras. It is observed in [14] that an L–fuzzy subset μ of A is an L–fuzzy ideal of A if and only if every α–level set of μ is an ideal of A. Here we define the commutator of L–fuzzy ideals using their level ideals.

Definition 12. The commutator of L–fuzzy ideals μ and σ of A denoted by $[\mu, \sigma]$ is an L–fuzzy subset of A defined by

$$[\mu, \sigma](x) = \bigvee \left\{\alpha \wedge \beta : \alpha, \beta \in L, x \in \left[\mu_\alpha, \sigma_\beta\right]\right\} \tag{9}$$

for all $x \in A$.

For each α, β, and λ in L with $\lambda = \alpha \wedge \beta$, the following can be verified.

$$x \in \left[\mu_\alpha, \sigma_\beta\right] \Longrightarrow x \in \left[\mu_\lambda, \sigma_\lambda\right] \tag{10}$$

So the commutator of L–fuzzy ideals can be equivalently redefined as follows.

$$[\mu, \sigma](x) = \bigvee \{\lambda \in L : x \in [\mu_\lambda, \sigma_\lambda]\} \tag{11}$$

The following lemmas can be verified easily.

Lemma 13. *For any L–fuzzy ideals μ and σ of A, $[\mu, \sigma]$ is an L–fuzzy ideal of A such that*

$$[\mu, \sigma] \le \mu \cap \sigma. \tag{12}$$

Lemma 14. *For any ideals I and J of A*

$$\chi_{[I,J]} = [\chi_I, \chi_J]. \tag{13}$$

In the following theorem, we give an algebraic characterization for the commutator of L–fuzzy ideals.

Theorem 15. *For each $x \in A$ and L–fuzzy ideals μ and σ of A*

$$[\mu,\sigma](x) = \bigvee \left\{ \mu^m(\vec{b}) \wedge \sigma^k(\vec{c}) : x = t(\vec{a},\vec{b},\vec{c}),\ \text{where } \vec{a} \in A^n, \vec{b} \in A^m, \vec{c} \in A^k,\ \text{and } t(\vec{x},\vec{y},\vec{z}) \text{ is a commutator term in } \vec{y},\vec{z} \right\}. \tag{14}$$

Proof. For each $x \in A$, let us define two sets H_x and G_x as follows.

$$H_x = \left\{ \mu^m(\vec{b}) \wedge \sigma^k(\vec{c}) : x = t(\vec{a},\vec{b},\vec{c}),\ \text{where } \vec{a} \in A^n, \vec{b} \in A^m, \vec{c} \in A^k \text{ and } t(\vec{x},\vec{y},\vec{z}) \text{ is a commutator term in } \vec{y},\vec{z} \right\} \tag{15}$$

$$G_x = \{\alpha \in L : x \in [\mu_\alpha, \sigma_\alpha]\}$$

Clearly both H_x and G_x are subsets of L. Our claim is to see the following.

$$\bigvee \{\alpha : \alpha \in H_x\} = \bigvee \{\alpha : \alpha \in G_x\} \tag{16}$$

One way of proof is to show that $H_x \subseteq G_x$. $\alpha \in H_x$ implies that $\alpha = \mu^m(\vec{b}) \wedge \sigma^k(\vec{c})$, where $x = t(\vec{a},\vec{b},\vec{c})$ for some $\vec{a} \in A^n, \vec{b} \in A^m, \vec{c} \in A^k$, and some commutator term $t(\vec{x},\vec{y},\vec{z})$ in $\vec{y},\vec{z}$. That is, $\vec{b} \in (\mu_\alpha)^m$ and $\vec{c} \in (\sigma_\alpha)^k$ so that $x \in [\mu_\alpha, \sigma_\alpha]$. Then $\alpha \in G_x$ and hence $H_x \subseteq G_x$. Another way is to prove that, for each $\alpha \in G_x$, there exists $\beta \in H_x$ such that $\alpha \leq \beta$. If $\alpha \in G_x$, then $x \in [\mu_\alpha, \sigma_\alpha]$ so that $x = t(\vec{a},\vec{b},\vec{c})$ for some $\vec{a} \in A^n$, $\vec{b} \in (\mu_\alpha)^m$, and $\vec{c} \in (\sigma_\alpha)^k$, where $t(\vec{x},\vec{y},\vec{z})$ is a commutator term in $\vec{y},\vec{z}$. That is, $\mu^m(\vec{b}) \geq \alpha$ and $\sigma^k(\vec{c}) \geq \alpha$. If we put $\beta = \mu^m(\vec{b}) \wedge \sigma^k(\vec{c})$, then $\beta \geq \alpha$ and $\beta \in H_x$. This completes the proof. □

Notation 16. We write $F \subset\subset A$, to say that F is a finite subset of A.

Theorem 17. *For each $x \in A$ and L–fuzzy ideals μ and σ of A*

$$[\mu,\sigma](x) = \bigvee \left\{ \bigwedge_{a\in E, b\in F} (\mu(a) \wedge \sigma(b)) : x \in [E,F], E, F \subset\subset A \right\}. \tag{17}$$

Proof. For each $x \in A$, let us take the set G_x as in Theorem 15 and define a set H_x as follows.

$$H_x = \left\{ \bigwedge_{a\in E, b\in F} (\mu(a) \wedge \sigma(b)) : x \in [E,F], E, F \subset\subset A \right\} \tag{18}$$

Our claim is to show the following.

$$\bigvee \{\alpha : \alpha \in H_x\} = \bigvee \{\alpha : \alpha \in G_x\} \tag{19}$$

One way of proof is to show that $H_x \subseteq G_x$. If $\alpha \in H_x$, then

$$\alpha = \bigwedge_{a\in E, b\in F} (\mu(a) \wedge \sigma(b)) \tag{20}$$

where E and F are finite subsets of A such that $x \in [E,F]$. That is, $\mu(a) \wedge \sigma(b) \geq \alpha$ for all $a \in E$ and all $b \in F$. Then $E \subseteq \mu_\alpha$ and $F \subseteq \sigma_\alpha$ so that $[E,F] \subseteq [\mu_\alpha, \sigma_\alpha]$. Thus $x \in [\mu_\alpha, \sigma_\alpha]$ and hence $\alpha \in G_x$. Therefore $H_x \subseteq G_x$. Another way is to prove that, for each $\alpha \in G_x$, there exists $\beta \in H_x$ such that $\alpha \leq \beta$. If $\alpha \in G_x$, then $x \in [\mu_\alpha, \sigma_\alpha]$. Therefore, $x = t(\vec{a},\vec{b},\vec{c})$ for some $\vec{a} \in A^n$, $\vec{b} = \langle b_1, b_2, \ldots, b_m\rangle \in (\mu_\alpha)^m$, and $\vec{c} = \langle c_1, c_2, \ldots, c_k\rangle \in (\sigma_\alpha)^k$, where $t(\vec{x},\vec{y},\vec{z})$ is a commutator term in $\vec{y},\vec{z}$. That is,

$$\mu^m(\vec{b}) = \mu(b_1) \wedge \cdots \wedge \mu(b_m) \geq \alpha \tag{21}$$

and

$$\sigma^k(\vec{c}) = \sigma(c_1) \wedge \cdots \wedge \sigma(c_m) \geq \alpha. \tag{22}$$

If we put $E = \{b_1, b_2, \ldots, b_m\}$ and $F = \{c_1, c_2, \ldots, c_k\}$, then E and F are both finite subsets of A such that $x \in [E,F]$. Moreover, if we take

$$\beta = \bigwedge_{a\in E, b\in F} (\mu(a) \wedge \sigma(b)) \tag{23}$$

then $\beta \in H_x$ such that $\alpha \leq \beta$. This completes the proof. □

Definition 18. For each $\mu \in \mathcal{F}\mathcal{I}(A)$, we define by induction

$$\mu^{(1)} = \mu = \mu^1;$$
$$\mu^{(n+1)} = [\mu^{(n)}, \mu^{(n)}] \tag{24}$$
$$\text{and } \mu^{n+1} = [\mu^{(n)}, \mu].$$

An L–fuzzy ideal μ of A will be called fuzzy nilpotent (resp., fuzzy solvable) if $\mu^n = \chi_{(0)}$ (resp., $\mu^{(n)} = \chi_{(0)}$) for some $n \in Z_+$.

Lemma 19. *An L–fuzzy subset μ of A is L–fuzzy nilpotent (resp., L–fuzzy solvable) if and only if μ_α is nilpotent (resp., solvable) for all $\alpha \in L$.*

4. L–Fuzzy Prime Ideals

In this section we define L–fuzzy prime ideals and investigate some of their properties.

Definition 20. A nonconstant L–fuzzy ideal μ of A is called an L–fuzzy prime ideal if and only if

$$[\nu, \sigma] \leq \mu \Longrightarrow \text{ either } \nu \leq \mu \text{ or } \sigma \leq \mu \tag{25}$$

for all $\nu, \sigma \in \mathscr{F}\mathscr{I}(A)$.

Notation 21. For L–fuzzy points x_α and y_β of A, we denote $[\langle x_\alpha\rangle, \langle y_\beta\rangle]$ by $[x_\alpha, y_\beta]$.

In the following theorem we characterize L–fuzzy prime ideals using L–fuzzy points.

Theorem 22. *A nonconstant L–fuzzy ideal μ of A is L–fuzzy prime if and only if for any L–fuzzy points x_α and y_β of A*

$$\left[x_\alpha, y_\beta\right] \leq \mu \text{ either } x_\alpha \in \mu \text{ or } y_\beta \in \mu. \tag{26}$$

Proof. Suppose that μ satisfies the condition:

$$\left[x_\alpha, y_\beta\right] \leq \mu \Longrightarrow \text{ either } x_\alpha \in \mu \text{ or } y_\beta \in \mu \tag{27}$$

for all L–fuzzy points x_α and y_β of A. Let σ and θ be L–fuzzy ideals of A such that $[\sigma, \theta] \leq \mu$. Suppose if possible that $\sigma \nleq \mu$ and $\theta \nleq \mu$. Then there exist $x, y \in A$ such that $\sigma(x) \nleq \mu(x)$ and $\theta(y) \nleq \mu(y)$. If we put $\alpha = \sigma(x)$ and $\beta = \theta(y)$, then x_α and y_β are fuzzy points of A such that $x_\alpha \in \sigma$, but $x_\alpha \notin \mu$, and $y_\beta \in \theta$, but $y_\beta \notin \mu$, so that $[x_\alpha, y_\beta] \leq [\sigma, \theta] \leq \mu$, but $x_\alpha \notin \mu$ and $y_\beta \notin \mu$. This contradicts our hypothesis. Thus either $\sigma \leq \mu$ or $\theta \leq \mu$. Therefore μ is prime. The other way is clear. □

Theorem 23. *A nonconstant L–fuzzy ideal μ is an L–fuzzy prime ideal if and only if $Img(\mu) = \{1, \alpha\}$, where α is a prime element in L and the set $\mu_* = \{x \in A : \mu(x) = 1\}$ is a prime ideal of A.*

Proof. Suppose that μ is a prime L–fuzzy ideal. Clearly $1 \in Img(\mu)$, and since μ is nonconstant, there is some $a \in A$ such that $\mu(a) < 1$. We show that $\mu(a) = \mu(b)$ for all $a, b \in A - \mu_*$. Let $a, b \in A$ such that $\mu(a) < 1$ and $\mu(b) < 1$. Let us define L–fuzzy subsets σ and θ of A as follows:

$$\sigma(x) = \begin{cases} 1 & if\ \ x \in \langle a\rangle \\ 0 & otherwise \end{cases} \tag{28}$$

and

$$\theta(x) = \begin{cases} 1 & if\ \ x = 0 \\ \mu(a) & otherwise \end{cases} \tag{29}$$

for all $x \in A$. Then it can be verified that both σ and θ are L–fuzzy ideals of A. Moreover, for each $x \in A$ we have

$$[\sigma, \theta](x) = \begin{cases} 1 & if\ \ x = 0 \\ \mu(a) & if\ \ x \in \langle a\rangle - \{0\} \\ 0 & otherwise \end{cases} \tag{30}$$

so that $[\sigma, \theta] \leq \mu$, but $\sigma(a) = 1 > \mu(b)$, so $\sigma \nleq \mu$. Since μ is L–fuzzy prime, we get $\theta \leq \mu$, which gives $\theta(b) \leq \mu(b)$; that is, $\mu(a) \leq \mu(b)$. Similarly it can be verified that $\mu(b) \leq \mu(a)$ so that $\mu(a) = \mu(b)$ for all $a, b \in A - \mu_*$. Thus $Img(\mu) = \{1, \alpha\}$ for some $\alpha \neq 1$ in L. It remains to show that α is a prime element in L. Let $\beta, \gamma \in L$ such that $\beta \wedge \gamma \leq \alpha$. Consider L–fuzzy subsets $\overline{\beta}$ and $\overline{\gamma}$ of A defined by

$$\overline{\beta}(x) = \begin{cases} 1 & if\ \ x = 0 \\ \beta & otherwise \end{cases} \tag{31}$$

and

$$\overline{\gamma}(x) = \begin{cases} 1 & if\ \ x = 0 \\ \gamma & otherwise \end{cases} \tag{32}$$

for all $x \in A$. Then $\overline{\beta}$ and $\overline{\gamma}$ are both L–fuzzy ideals of A such that $[\overline{\beta}, \overline{\gamma}] \leq \mu$. Since μ is L–fuzzy prime, either $\overline{\beta} \leq \mu$ or $\overline{\gamma} \leq \mu$ so that either $\beta \leq \alpha$ or $\gamma \leq \alpha$. Hence α is prime in L. Next we show that the level ideal μ_* is prime. Put $P = \mu_*$ and let I and J be ideas of A such that $[I, J] \subseteq P$. Then $\chi_{[I,J]} \leq \chi_P \leq \mu$. That is, $[\chi_I, \chi_J] \leq \mu$. Since μ is L–fuzzy prime, either $\chi_I \leq \mu$ or $\chi_J \leq \mu$, implying that either $I \subseteq P$ or $J \subseteq P$. Therefore P is prime. Conversely suppose that $Img(\mu) = \{1, \alpha\}$, where α is a prime element in L and $P = \mu_*$ is a prime ideal of A. Let σ and θ be L–fuzzy ideals of A such that $[\sigma, \theta] \leq \mu$. Suppose if possible that there exist $x, y \in A$ such that $\sigma(x) \nleq \mu(x)$ and $\theta(y) \nleq \mu(y)$. Since μ is 2–valued, $\mu(x) = \mu(y) < 1$ so that both x and y do not belong to P. Since P is prime, there exists $z \in [x, y]$ such that $z \notin P$; that is, $\mu(z) = \alpha$. Otherwise, if $[x, y] \subseteq P$, then either $x \in P$ or $y \in P$. As $z \in [x, y]$, $z = t(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})$ for some $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in \langle x\rangle^m$, $\overrightarrow{c} \in \langle y\rangle^k$, where $t(\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$ is a commutator term in $\overrightarrow{v}, \overrightarrow{w}$. Now consider the following.

$$\begin{aligned} \alpha = \mu(z) \geq [\sigma, \theta](z) \geq \sigma^m\left(\overrightarrow{b}\right) \wedge \theta^k\left(\overrightarrow{c}\right) \\ \geq \sigma(x) \wedge \theta(y) \end{aligned} \tag{33}$$

That is, $\sigma(x) \wedge \theta(y) \leq \alpha$. Since α is a prime element in L, it follows that either $\sigma(x) \leq \alpha = \mu(x)$ or $\theta(y) \leq \alpha = \mu(y)$, which is a contradiction. Therefore μ is L–fuzzy prime. □

Let P be a prime ideal of A and α be a prime element in L. Consider an L–fuzzy subset α_P of A defined by

$$\alpha_P(x) = \begin{cases} 1 & if\ \ x \in P \\ \alpha & otherwise \end{cases} \tag{34}$$

for all $x \in A$. The above theorem confirms that L−fuzzy prime ideals of A are only of the form α_P. This establishes a one-to-one correspondence between the class of all L−fuzzy prime ideals of A and the collection of all pairs (P, α) where P is a prime ideal in A and α is a prime element in L.

Corollary 24. *Let P be an ideal of A and α a prime element in L. Then P is a prime ideal if and only if α_P is an L−fuzzy prime ideal.*

Theorem 25. *If μ is an L−fuzzy prime ideal of A, then*

$$\mu(a) \vee \mu(b) \geq \bigwedge \{\mu(x) : x \in [a, b]\} \tag{35}$$

for all $a, b \in A$.

Proof. We use proof by contradiction. Suppose if possible the following.

$$\bigwedge \{\mu(x) : x \in [a, b]\} \nleq \mu(a) \vee \mu(b) \tag{36}$$

Then $\mu(x) \nleq \mu(a) \vee \mu(b)$ for all $x \in [a, b]$. Since μ is prime, by Theorem 23 there exists a prime element $\alpha < 1$ in L such that $\mu(a) = \alpha = \mu(b)$ and $\mu(x) = 1$ for all $x \in [a, b]$, so $[a, b] \subseteq \mu_*$. Since μ_* is a prime ideal of A (see Theorem 23), we get that either $a \in \mu_*$ or $b \in \mu_*$. This is a contradiction. Thus the result holds. □

Theorem 26. *If $Img(\mu) = \{1, \alpha\}$, where α is a prime element in L and μ satisfies the condition:*

$$\mu(a) \vee \mu(b) \geq \bigwedge \{\mu(x) : x \in [a, b]\} \tag{37}$$

for all $a, b \in A$, then μ is L−fuzzy prime.

It is natural to ask ourselves, does every algebra in $\mathscr{K}$ have L−fuzzy prime ideals? Of course, probably no. In the following theorem we give a sufficient condition for an algebra A to have L−fuzzy prime ideals.

Theorem 27. *Let A be an algebra satisfying the following.*

$$x \in [x, x] \quad \text{for all } x \in A \tag{38}$$

If $a \in A$ and μ is an L−fuzzy ideal of A such that $\mu(a) \leq \alpha$ where α is an irreducible element in L, then there exists an L−fuzzy prime ideal θ of A such that

$$\begin{gathered} \mu \leq \theta \\ \text{and } \theta(a) \leq \alpha. \end{gathered} \tag{39}$$

Proof. Put $\mathfrak{F} = \{\sigma \in \mathscr{F}\mathscr{I}(A) : \mu \leq \sigma \text{ and } \sigma(a) \leq \alpha\}$. Clearly $\mu \in \mathfrak{F}$ so that $\mathfrak{F}$ is nonempty and hence it forms a poset under the inclusion ordering of L−fuzzy sets. By applying Zorn's lemma we can choose a maximal element, say θ, in $\mathfrak{F}$. Now it is enough to show that θ is prime. Suppose not. Then there exist L−fuzzy ideals σ and ν of A such that $[\sigma, \nu] \leq \theta$ but $\sigma \nleq \theta$ and $\nu \nleq \theta$. Put $\theta_1 = \theta \vee \sigma$ and $\theta_2 = \theta \vee \nu$. Then θ_1 and θ_2 are L−fuzzy ideals of A such that $\theta \subsetneqq \theta_1$ and $\theta \subsetneqq \theta_2$. By the maximality of θ in $\mathfrak{F}$ both θ_1 and θ_2 do not belong to $\mathfrak{F}$. Thus

$$\begin{gathered} \theta_1(a) \nleq \alpha \\ \text{and } \theta_2(a) \nleq \alpha. \end{gathered} \tag{40}$$

Since $a \in [a, a]$ and α is $\wedge$−irreducible element in L, we get $[\theta_1, \theta_2](a) \nleq \alpha$, so $\theta(a) \nleq \alpha$, which is a contradiction. Therefore θ is prime. □

If A is a nontrivial algebra such that $a \in [a, a]$ for all $a \in A$, then it can be deduced from the above theorem that L−fuzzy prime ideals exist in A.

Definition 28. An L−fuzzy subset λ of A is called an L−fuzzy m−system (resp., an L−fuzzy n−system) if for all $a, b \in A$, there exists $x \in [a, b]$ (resp., there exists $x \in [a, a]$) such that

$$\lambda(x) \geq \lambda(a) \wedge \lambda(b) \quad (\textit{resp. } \lambda(x) \geq \lambda(a)). \tag{41}$$

Lemma 29. *An L−fuzzy subset λ of A is an L−fuzzy m−system (resp., a fuzzy n−system) if and only if the level set λ_α is an m−system (resp., an n−system) for all $\alpha \in L$.*

Theorem 30. *Let μ be an L−fuzzy ideal of A and λ an L−fuzzy m−system such that $\mu \cap \lambda \leq \alpha$, where α is an irreducible element in L. Then there exists L−fuzzy prime ideal θ of A such that*

$$\begin{gathered} \mu \leq \theta \\ \text{and } \theta \cap \lambda \leq \alpha. \end{gathered} \tag{42}$$

Proof. Put $\mathfrak{F} = \{\sigma \in \mathscr{F}\mathscr{I}(A) : \mu \leq \sigma \text{ and } \sigma \cap \lambda \leq \alpha\}$. Clearly $\mu \in \mathfrak{F}$ so that $\mathfrak{F}$ is nonempty, and hence it forms a poset under the inclusion ordering of L−fuzzy sets. By applying Zorn's lemma we can choose a maximal element, say θ, in $\mathfrak{F}$. Now it is enough to show that θ is prime. Suppose not. Then there exist L−fuzzy ideals σ and ν of A such that $[\sigma, \nu] \leq \theta$, but $\sigma \nleq \theta$ and $\nu \nleq \theta$. Put $\theta_1 = \theta \vee \sigma$ and $\theta_2 = \theta \vee \nu$. Then θ_1 and θ_2 are L−fuzzy ideals of A such that $\theta \subsetneqq \theta_1$ and $\theta \subsetneqq \theta_2$. By the maximality of θ in $\mathfrak{F}$ both θ_1 and θ_2 do not belong to $\mathfrak{F}$ so there exist $a, b \in A$ such that

$$\begin{gathered} (\theta_1 \cap \lambda)(a) \nleq \alpha \\ \text{and } (\theta_2 \cap \lambda)(b) \nleq \alpha. \end{gathered} \tag{43}$$

Since $[\sigma, \nu] \leq \theta$, we have $[\theta_1, \theta_2] \leq \theta$. If $x \in [a, b]$, then $x = t(\overrightarrow{c}, \overrightarrow{u}, \overrightarrow{v})$ for some $\overrightarrow{c} \in A^n$, $\overrightarrow{u} \in \langle a \rangle^m$, $\overrightarrow{v} \in \langle b \rangle^k$, and some commutator term $t(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z})$ in $\overrightarrow{y}, \overrightarrow{z}$. Then for each $x \in [a, b]$ the following holds.

$$\theta(x) \geq [\theta_1, \theta_2](x) \geq \theta_1(a) \wedge \theta_2(b) \tag{44}$$

Also we have $\theta_1(a) \wedge \theta_2(b) \nleq \alpha$, which gives that $\theta(x) \nleq \alpha$. However, since $\theta \cap \lambda \leq \alpha$ and α is an irreducible element in L, we get that $\lambda(x) \leq \alpha$ for all $x \in [a, b]$. This contradicts that λ is a fuzzy m−system. Therefore θ is prime. □

For a nontrivial algebra A, to have an L−fuzzy m−system is a sufficient condition for A to possess L−fuzzy prime ideals.

5. Maximal Fuzzy L−Ideals

A maximal L−fuzzy ideal of A is a maximal element in the collection of all nonconstant L−fuzzy ideals of A under the pointwise partial ordering of L−fuzzy sets.

An element $1 \neq \alpha$ in L is called a dual atom if there is no β in L such that $\alpha < \beta < 1$. In other words α is maximal in $L - \{1\}$. In the following theorem we give an internal characterization of L–fuzzy maximal ideals in A.

Theorem 31. *An L–fuzzy ideal μ of A is maximal if and only if $Img(\mu) = \{1, \alpha\}$, where α is a dual atom in L and the set $\mu_* = \{x \in A : \mu(x) = 1\}$ is a maximal ideal of A.*

Proof. Suppose that μ is maximal. Clearly $1 \in Img(\mu)$, and since μ is nonconstant, there is some $a \in A$ such that $\mu(a) < 1$. We first show that μ assumes exactly one value other than 1. Let $x, y \in A$ such that $\mu(x) < 1$ and $\mu(y) < 1$. Put $\alpha = \mu(x)$ and $\beta = \mu(y)$. Define L–fuzzy subsets $\mu_\alpha^\vee$ and $\mu_\beta^\vee$ of A as follows:

$$\begin{aligned} \mu_\alpha^\vee(z) &= \mu(z) \vee \alpha \\ \text{and } \mu_\beta^\vee(z) &= \mu(z) \vee \beta \end{aligned} \tag{45}$$

for all $z \in A$. Then it can be verified that both $\mu_\alpha^\vee$ and $\mu_\beta^\vee$ are L–fuzzy ideals of A such that $\mu \leq \mu_\alpha^\vee$ and $\mu \leq \mu_\beta^\vee$. By the maximality of μ we get that $\mu = \mu_\alpha^\vee$ and $\mu = \mu_\beta^\vee$. Thus $\alpha = \beta$. Therefore $Img(\mu) = \{1, \alpha\}$ for some $\alpha \in L - \{1\}$. Next we prove that this α is a dual atom. Let $\beta \in L$ such that $\alpha < \beta$. Define an L–fuzzy subset σ of A by

$$\sigma(z) = \begin{cases} 1 & if\ \mu(z) = 1 \\ \beta & otherwise \end{cases} \tag{46}$$

for all $z \in A$. Then σ is an L–fuzzy ideal of A such that $\mu < \sigma$. By the maximality of μ it yields that $\sigma = 1_A$; i.e., $\sigma(z) = 1$ for all $z \in A$, so $\beta = 1$. Therefore α is a dual atom. It remains to show that μ_* is a maximal ideal of A. Clearly it is a proper ideal. Let J be a proper ideal of A such that $\mu_* \subseteq J$. Define an L–fuzzy subset σ of A by

$$\sigma(z) = \begin{cases} 1 & if\ z \in J \\ \alpha & otherwise \end{cases} \tag{47}$$

for all $z \in A$. Then σ is a nonconstant L–fuzzy ideal of A such that $\mu \leq \sigma$. Since μ is maximal, we get $\mu = \sigma$, so $\mu_* = J$. Therefore μ_* is maximal among all proper ideals of A. Conversely suppose that $Img(\mu) = \{1, \alpha\}$, where α is a dual atom in L and the set $\mu_* = \{x \in A : \mu(x) = 1\}$ is a maximal ideal of A. Let σ be a nonconstant L–fuzzy ideal of A such that $\mu \leq \sigma$. Then $\sigma(x) = 1$ for all $x \in \mu_*$ and $\alpha \leq \sigma(x)$ for all $x \in A - \mu_*$. We show that $\sigma = \mu$. Suppose not. Then there exists $x \in A$ such that $\sigma(x) > \mu(x)$, so $x \in A - \mu_*$. If $\sigma(x) = 1$, then $x \in \sigma_* = \{z \in A : \sigma(z) = 1\}$ and $\mu_* \subsetneq \sigma_* \subsetneq A$. This contradicts the maximality of μ_*. Also if $\sigma(x) < 1$, then $\alpha \leq \sigma(x) < 1$. Again this contradicts the hypothesis that α is a dual atom. Therefore $\sigma = \mu$. Hence μ is maximal. □

The above theorem confirms that there is a one-to-one correspondence between the class of all maximal L–fuzzy ideals and the set of all pairs (M, α) where M is a maximal ideal in A and α is a dual atom in L.

Theorem 32. *If A is an algebra in which every maximal ideal is a prime ideal, then every maximal L–fuzzy ideal is an L–fuzzy prime ideal.*

For instance, if $[A, A] = A$, then every maximal L–fuzzy ideal is an L–fuzzy prime ideal.

6. Conclusion

In this paper, the commutator (or the product) of L–fuzzy ideals is defined and investigated in a more general context, in universal algebras. By the use of this commutator, L–fuzzy prime ideals in universal algebras are defined and fully characterized. Furthermore, maximal L–fuzzy ideals of universal algebra are studied.

The study of fuzzy semiprime ideals in universal algebras is under investigation by the authors using the commutator of L–fuzzy ideals. Moreover, the radical of L–fuzzy ideals in universal algebras will be studied and would be applied to characterize the central properties of L–fuzzy semiprime ideals.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This research work was partially supported by a research grant from the College of Science, Bahir Dar University.

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