Hindawi
Advances in Fuzzy Systems
Volume 2019, Article ID 4263923, 13 pages
https://doi.org/10.1155/2019/4263923

Research Article

Fuzzy Ideals and Fuzzy Filters of Pseudocomplemented Semilattices

Berhanu Assaye Alaba[1] and Wondwosen Zemene Norahun[2]

[1]*Department of Mathematics, Bahir Dar University, Bahir Dar, Ethiopia*
[2]*Department of Mathematics, University of Gondar, Gondar, Ethiopia*

Correspondence should be addressed to Wondwosen Zemene Norahun; wondie1976@gmail.com

Received 26 March 2019; Revised 13 May 2019; Accepted 17 June 2019; Published 8 July 2019

Academic Editor: Jose A. Sanz

In this paper, we introduce the concept of kernel fuzzy ideals and $*$-fuzzy filters of a pseudocomplemented semilattice and investigate some of their properties. We observe that every fuzzy ideal cannot be a kernel of a $*$-fuzzy congruence and we give necessary and sufficient conditions for a fuzzy ideal to be a kernel of a $*$-fuzzy congruence. On the other hand, we show that every fuzzy filter is the cokernel of a $*$-fuzzy congruence. Finally, we prove that the class of $*$-fuzzy filters forms a complete lattice that is isomorphic to the lattice of kernel fuzzy ideals.

1. Introduction

The theory of pseudocomplementation was introduced and extensively studied in semilattices and particularly in distributive lattices by O. Frink [1] and G. Birkhoff [2]. Later, pseudocomplement in Stone algebra has been studied by several authors like R. Balbes [3], G. Grätzer [4], etc. In 1973, W. H. Cornish [5] studied congruence on pseudocomplemented distributive lattices and identified those ideals and filters that are congruence kernels and cokernels, respectively. Later, T. S. Blyth [6] studied ideals and filters of pseudocomplemented semilattices.

On the other hand, the concept of fuzzy sets was firstly introduced by Zadeh [7]. Rosenfeld has developed the concept of fuzzy subgroups [8]. Since then, several authors have developed interesting results on fuzzy theory; see [8–19].

In this paper, we introduce the concept of kernel fuzzy ideals and $*$-fuzzy filters of a pseudocomplemented semilattice. We studied a $*$-fuzzy congruence on a pseudocomplemented semilattice. We observe that every fuzzy ideal cannot be a kernel of a $*$-fuzzy congruence. We give necessary and sufficient conditions for a fuzzy ideal to be a kernel of a $*$-fuzzy congruence. We also show that the class of kernel fuzzy ideals can be made a complete distributive lattice. Moreover, we study the image and preimage of kernel fuzzy ideals under a $*$-epimorphism mapping. In Section 4 we turn our attention to fuzzy filters. Here we show that every fuzzy filter is the cokernel of a $*$-fuzzy congruence and investigate a certain type of fuzzy filter called a $*$-fuzzy filter. We prove that these filters form a complete lattice that is isomorphic to the lattice of kernel fuzzy ideals.

2. Preliminaries

In this section, we recall some definitions and basic results on lattices, meet semilattices, and fuzzy theory.

Definition 1 (see [2]). An algebra $(L, \wedge, \vee)$ is said to be a lattice if it satisfies the following conditions:

(1) $x \wedge x = x$ and $x \vee x = x$,

(2) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$,

(3) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$,

(4) $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = y$.

Definition 2 (see [2]). A lattice $(L, \wedge, \vee)$ is a distributive lattice if and only if it satisfies the following identity:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z). \quad (1)$$

A lattice L is said to be bounded if there exist 0 and 1 in L such that $0 \wedge x = 0$ and $1 \vee x = 1$ for all $x \in L$.

If L and M are lattices, then $f : L \longrightarrow M$ is said to be a lattice morphism if for all $x, y \in L$.

(1) $f(x \wedge y) = f(x) \wedge f(y)$

(2) $f(x \vee y) = f(x) \vee f(y)$.

If f is onto, then f is an epimorphism.

Definition 3 (see [20]). An algebra $(L, \wedge, \vee, *, 0, 1)$ is a pseudocomplemented lattice if the following conditions hold:

(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,

(2) for all $x, y \in L, x \wedge y = 0 \Longleftrightarrow x \wedge y^* = x$.

If $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice, then $(L, \wedge, \vee, *, 0, 1)$ is a pseudocomplemented distributive lattice.

The lattice $(L, \wedge, \vee, 0, 1)$ is said to be a complemented lattice if satisfies the following conditions for all $x \in L$ there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' = 1$.

Definition 4 (see [3]). A pseudocomplemented distributive lattice L is called a Stone algebra if, for all $x \in L$, it satisfies the property:

$$x^* \vee x^{**} = 1. \quad (2)$$

Definition 5 (see [1]). An algebra $(S, \wedge, *, 0)$ is a pseudocomplemented semilattice if for all $x, y, z \in S$ the following conditions are satisfied:

(1) $x \wedge x = x$,

(2) $x \wedge y = y \wedge x$,

(3) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,

(4) $x \wedge y = 0 \Longleftrightarrow x \wedge y^* = x$.

It is well known that S is a partially ordered set relative to the order relation defined by $x \leq y \Longleftrightarrow x \wedge y = x$ and that relative to this order relation $x \wedge y$ is the greatest lower bound of x and y. Hence the condition of Definition 5(4) is equivalent to

$$x \wedge y = 0 \Longleftrightarrow x \leq y^*. \quad (3)$$

A nonempty subset I of a semilattice S is called an ideal of S if, $y \in I, x \in S \Longrightarrow y \wedge x \in I$.

A nonempty subset F of a semilattice S is called a filter of S if it satisfies the following:

(1) $x, y \in F \Longrightarrow x \wedge y \in F$,

(2) $y \in F, x \in S, y \leq x \Longrightarrow x \in F$.

Theorem 6 (see [1]). *For any two elements x, y of a pseudocomplemented semilattice S, we have the following:*

(1) $0^{**} = 0$,

(2) $x \wedge (x \wedge y)^* = x \wedge y^*$,

(3) $x \wedge x^* = 0$,

(4) $x \leq y \Longrightarrow y^* \leq x^*$,

(5) $x \leq x^{**}$,

(6) $x^{***} = x^*$,

(7) $x^{****} = x^{**}$,

(8) $(x \wedge y)^{**} = x^{**} \wedge y^{**}$,

(9) $(x \wedge y)^* = (x^{**} \wedge y^{**})^*$.

An element x of a pseudocomplemented semilattice is called closed if $x = x^{**}$.

Definition 7 (see [7]). Let X be any nonempty set. A mapping $\mu : X \longrightarrow [0, 1]$ is called a fuzzy subset of X.

The unit interval $[0, 1]$ together the operations min and max form a complete distributive lattice. We often write $\wedge$ for minimum or infimum and $\vee$ for maximum or supremum. That is, for all $\alpha, \beta \in [0, 1]$ we have $\alpha \wedge \beta = \min\{\alpha, \beta\}$ and $\alpha \vee \beta = \max\{\alpha, \beta\}$.

Definition 8 (see [8]). Let μ and θ be fuzzy subsets of a set A. Define the fuzzy subsets $\mu \cup \theta$ and $\mu \cap \theta$ of A as follows: for each $x \in A$,

$$\begin{aligned} (\mu \cup \theta)(x) &= \mu(x) \vee \theta(x), \\ (\mu \cap \theta)(x) &= \mu(x) \wedge \theta(x). \end{aligned} \quad (4)$$

Then $\mu \cup \theta$ and $\mu \cap \theta$ are called the union and intersection of μ and θ, respectively.

For any collection, $\{\mu_j : j \in J\}$ of fuzzy subsets of X, where J is a nonempty index set, the least upper bound $\bigcup_{j \in J} \mu_j$, and the greatest lower bound $\bigcap_{j \in J} \mu_j$ of the μ_j's are given by for each $x \in X$,

$$\begin{aligned} \left(\bigcup_{j \in J} \mu_j\right)(x) &= \bigvee_{j \in J} \mu_j(x), \\ \left(\bigcap_{j \in J} \mu_j\right)(x) &= \bigwedge_{j \in J} \mu_j(x), \end{aligned} \quad (5)$$

respectively.

For each $t \in [0, 1]$ the set

$$\mu_t = \{x \in A : \mu(x) \geq t\} \quad (6)$$

is called the level subset of μ at t [7].

The characteristics function of any set A is defined as

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \quad (7)$$

Definition 9 (see [8]). Let f be a function from X into Y; μ be a fuzzy subset of X; and θ be a fuzzy subset of Y. The image of μ under f, denoted by $f(\mu)$, is a fuzzy subset of Y such that

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{otherwise,} \end{cases} \quad \text{where } y \in Y. \tag{8}$$

The preimage of θ under f, symbolized by $f^{-1}(\theta)$, is a fuzzy subset of X and

$$f^{-1}(\theta)(x) = \theta(f(x)) \quad \text{for all } x \in X. \tag{9}$$

Definition 10 (see [18]). A fuzzy subset μ of a bounded lattice L is called a fuzzy ideal of L if for all $x, y \in L$ the following conditions are satisfied:

(1) $\mu(0) = 1$,

(2) $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$,

(3) $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$.

Definition 11 (see [18]). A fuzzy subset μ of a bounded lattice L is called a fuzzy filter of L if for all $x, y \in L$ the following conditions are satisfied:

(1) $\mu(1) = 1$,

(2) $\mu(x \vee y) \geq \mu(x) \vee \mu(y)$,

(3) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$.

Theorem 12 (see [21]). *Let L be a lattice, $x \in L$, and $\alpha \in [0,1]$. Define a fuzzy subset α_x of L as*

$$\alpha_x(y) = \begin{cases} 1, & \text{if } y \leq x \\ \alpha, & \text{if } y \nleq x \end{cases} \tag{10}$$

is a fuzzy ideal of L.

Remark 13 (see [21]). α_x is called the α-level principal fuzzy ideal corresponding to x.

Similarly, a fuzzy subset α^x of L defined as

$$\alpha^x(y) = \begin{cases} 1, & \text{if } x \leq y \\ \alpha, & \text{if } x \nleq y \end{cases} \tag{11}$$

is the α-level principal fuzzy filter corresponding to x.

Definition 14 (see [18]). A fuzzy subset θ of $L \times L$ is said to be a fuzzy congruence on L if and only if, for any $x, y, z \in L$, the following hold:

(1) $\theta(x,x) = 1$,

(2) $\theta(x,y) = \theta(y,x)$,

(3) $\theta(x,y) \wedge \theta(y,z) \leq \theta(x,z)$,

(4) $\theta(x,y) \leq \theta(x \vee z, y \vee z) \wedge \theta(x \wedge z, y \wedge z)$.

3. Kernel Fuzzy Ideals and $*$-Fuzzy Ideals

In this section, we introduce the concept of kernel fuzzy ideals and $*$-fuzzy ideals of a pseudocomplemented semilattice. We give necessary and sufficient conditions for fuzzy ideals to be a kernel of a $*$-fuzzy congruence.

Throughout the rest of this paper, S stands for a pseudocomplemented semilattice $(S, \wedge, *, 0, 1)$ and a partially ordered set unless it is specified.

Now we define a fuzzy ideal of a semilattice.

Definition 15. A fuzzy subset μ of S is called a fuzzy ideal of S if for all $x, y \in S$ the following conditions are satisfied:

(1) $\mu(0) = 1$,

(2) $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$.

Theorem 16. *A fuzzy subset μ of S is a fuzzy ideal if and only if each level subset of μ is an ideal of S. (In particular, a nonempty subset I of S is an ideal of S if and only if χ_I is a fuzzy ideal.)*

Proof. Suppose μ is a fuzzy ideal of S. Let $x \in \mu_t$ and $y \in S, t \in [0,1]$. Then $0 \in \mu_t$ and $\mu(x) \vee \mu(y) \geq t$. Since μ is a fuzzy ideal, we get that $\mu(x \wedge y) \geq t$. Thus $x \wedge y \in \mu_t$. So μ_t is an ideal of S.

Conversely, suppose that every level subset of μ is a fuzzy ideal of S. Then μ_1 is a fuzzy ideal of S. Thus $\mu(0) = 1$. Now we proceed to show that $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$; let $x, y \in S$ such that $\mu(x) = \alpha$ and $\mu(y) = \beta$. Then either $\alpha \leq \beta$ or $\beta \leq \alpha$. Without loss of generality, $\alpha \leq \beta$. Since μ_β is an ideal of S and $y \in \mu_\beta$, then $x \wedge y \in \mu_\beta$. Thus $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$. So μ is a fuzzy ideal of S. □

Definition 17. A fuzzy subset θ of $S \times S$ is said to be a fuzzy congruence on S if and only if, for any $x, y, z, w \in S$, the following hold:

(1) $\theta(x,x) = 1$,

(2) $\theta(x,y) = \theta(y,x)$,

(3) $\theta(x,y) \wedge \theta(y,z) \leq \theta(x,z)$,

(4) $\theta(x,y) \wedge \theta(z,w) \leq \theta(x \wedge z, y \wedge w)$.

Definition 18. A fuzzy congruence relation θ on S is called a $*$-fuzzy congruence if $\theta(x,y) \leq \theta(x^*, y^*)$ for all $x, y \in S$.

It is clear that a fuzzy congruence θ is a $*$-fuzzy congruence on S if and only if each level subset $\theta_t = \{(x,y) \in S \times S : \theta(x,y) \geq t,\ t \in [0,1]\}$ of θ is a $*$-congruence on S. Also, let λ be a congruence on S. Then λ is a $*$-congruence on S if and only if its characteristic function χ_λ is a $*$-fuzzy congruence on S.

Theorem 19. *A fuzzy congruence θ on S is a $*$-fuzzy congruence if and only if*

$$\theta(x^*, 1) \geq \theta(x, 0). \tag{12}$$

Proof. If θ is a $*$-fuzzy congruence on S, then by Definition 18 $\theta(x^*, 1) \geq \theta(x, 0)$. Suppose conversely that the condition holds. Let $x, y \in S$. Since θ is a fuzzy congruence on S,

$\theta(x, y) \leq \theta(x^* \wedge x, x^* \wedge y) = \theta(0, x^* \wedge y)$. By the assumption, we have $\theta(0, x^* \wedge y) \leq \theta(1, (x^* \wedge y)^*)$. Thus $\theta(x, y) \leq \theta(1, (x^* \wedge y)^*) \leq \theta(x^*, x^* \wedge (x^* \wedge y)^*) = \theta(x^*, x^* \wedge y^*)$. Similarly, $\theta(x, y) \leq \theta(x^* \wedge y^*, y^*)$. Since θ is a fuzzy congruence, then $\theta(x^*, y^*) \geq \theta(x^*, x^* \wedge y^*) \wedge \theta(x^* \wedge y^*, y^*)$. Thus $\theta(x^*, y^*) \geq \theta(x, y)$. So θ is a $*$-fuzzy congruence on S. □

If θ is a fuzzy congruence on S and $x \in S$, then the fuzzy subset θ_x of S defined by

$$\theta_x(y) = \theta(x, y) \quad \text{for all } y \in S \tag{13}$$

is called a fuzzy congruence class of S determined by θ and x. We thus have the following theorem.

Theorem 20. *If θ is a fuzzy congruence on S, then the fuzzy congruence class θ_0 of S determined by θ and 0 is a fuzzy ideal of S.*

Proof. Suppose θ is a fuzzy congruence on S. Then $\theta(0, 0) = 1$. Thus $\theta_0(0) = 1$. Again for any $x, y \in S$, $\theta_0(x) = \theta(x, 0) \wedge \theta(y, y)$. Since θ is a fuzzy congruence on S, we have $\theta_0(x) \leq \theta(x \wedge y, 0) = \theta_0(x \wedge y)$. Similarly, $\theta_0(y) \leq \theta_0(x \wedge y)$. Thus $\theta_0(x \wedge y) \geq \theta_0(x) \vee \theta_0(y)$. So θ_0 is a fuzzy ideal of S. □

Now we define a kernel fuzzy ideal of a semilattice.

Definition 21. A fuzzy ideal μ of S is called a kernel fuzzy ideal if $\mu = \theta_0$, where θ is a $*$-fuzzy congruence on S.

Example 22. Consider the semilattice S whose Hasse diagram is given in Figure 1.

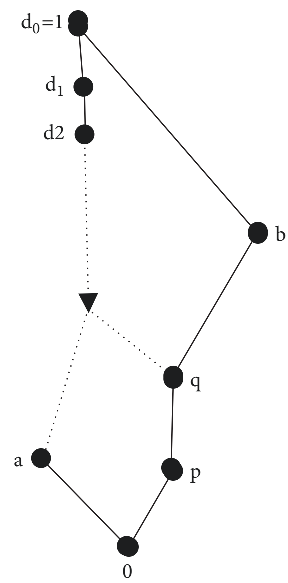


Figure 1

Then S is pseudocomplemented; we have $0^* = 1, a^* = b, p^* = q^* = b^* = a, d_i^* = 0$ for $i \geq 0$. The fuzzy ideal μ of S defined as $\mu(0) = 1, \mu(a) = \mu(p) = \mu(q) = \mu(b) = 0.5$, and $\mu(x) = 0$ for $x \neq 0, a, b, p, q$ is not the kernel fuzzy ideal of a $*$-fuzzy congruence. For, suppose that θ were a $*$-fuzzy congruence with kernel μ; then $\theta_0(1) \geq \theta(1, a) \wedge \theta(a, 0)$. Since $\theta(0, b) \leq \theta(0^*, b^*) = \theta(1, a)$, we have $\theta_0(1) \geq \theta(0, b) \wedge \theta(a, 0) = \mu(b) \wedge \mu(a) = 0.5$. This is a contradiction.

In Example 22 we observe that every fuzzy ideal of a pseudocomplemented semilattice S is not a kernel fuzzy ideal. In the following theorem we give the necessary and sufficient conditions for a fuzzy ideal of a pseudocomplemented semilattice to be a kernel of a $*$-fuzzy congruence.

Theorem 23. *A fuzzy ideal μ of S is a kernel fuzzy ideal if and only if $\mu((x^* \wedge y^*)^*) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in S$.*

Proof. Let μ be a kernel fuzzy ideal of a $*$-fuzzy congruence θ on S. Then $\mu = \theta_0$. Suppose $x, y \in S$. Then by the assumption $\theta(x^*, 1) \geq \theta(x, 0) = \mu(x)$ and $\theta(y^*, 1) \geq \mu(y)$. Thus $\mu(x) \wedge \mu(y) \leq \theta(x^*, 1) \wedge \theta(y^*, 1)$. Since θ is a fuzzy congruence on S, we have

$$\begin{aligned} \theta(x^*, 1) \wedge \theta(y^*, 1) &\leq \theta(x^* \wedge y^*, 1) \\ &\leq \theta((x^* \wedge y^*)^*, 0) \\ &= \mu((x^* \wedge y^*)^*) \end{aligned} \tag{14}$$

Thus $\mu((x^* \wedge y^*)^*) \geq \mu(x) \wedge \mu(y)$.

Conversely, suppose that the condition holds. Consider the fuzzy relation θ_μ defined on S by

$$\theta_\mu(x, y) = \sup\{\mu(z) : x \wedge z^* = y \wedge z^*,\ z \in S\}. \tag{15}$$

Clearly, θ_μ is both reflexive and symmetric. It is also transitive; if $x, y \in S$, then

$$\begin{aligned} \theta_\mu(x, y) \wedge \theta_\mu(y, z) &= \sup\{\mu(w) : x \wedge w^* = y \wedge w^*\} \\ &\quad \wedge \sup\{\mu(u) : y \wedge u^* = z \wedge u^*\} = \sup\{\mu(w) \\ &\quad \wedge \mu(u) : x \wedge w^* = y \wedge w^*,\ y \wedge u^* = z \wedge u^*\} \end{aligned} \tag{16}$$

If $x \wedge w^* = y \wedge w^*, y \wedge u^* = z \wedge u^*$, then $x \wedge (w^* \wedge u^*)^{**} = z \wedge (w^* \wedge u^*)^{**}$. By the assumption, we have

$$\begin{aligned} \theta_\mu(x, y) \wedge \theta_\mu(y, z) &\leq \sup\{\mu((w^* \wedge u^*)^*) : x \\ &\quad \wedge (w^* \wedge u^*)^{**} = z \wedge (w^* \wedge u^*)^{**}\} \\ &\leq \sup\{\mu(k) : x \wedge k^* = z \wedge k^*,\ k \in L\} = \theta_\mu(x, z) \end{aligned} \tag{17}$$

So that θ_μ is a fuzzy equivalence relation on S. By the similar procedure it can be easily verified that θ_μ is a fuzzy congruence on S.

To show θ_μ is a $*$-fuzzy congruence on S, let $x \in S$. Then $\theta_\mu(x^*, 1) = \sup\{\mu(z) : x^* \wedge z^* = z^*\}$. Since S is a pseudocomplemented semilattice, $x^* \wedge z^* = z^* \Longleftrightarrow x \wedge z^* =$

$0 \Longleftrightarrow x \wedge z^{**} = x$. Thus $\theta_\mu(x^*, 1) = \sup\{\mu(z) : x \le z^{**}\}$. So that

$$\theta_\mu(x, 0) = \sup\{\mu(z) : x \wedge z^* = 0\} = \sup\{\mu(z) : x \le z^{**}\} \le \theta_\mu(x^*, 1). \tag{18}$$

It follows by Theorem 19 that θ_μ is a $*$-fuzzy congruence on S.

Now taking $x = y$ in the condition we obtain $\mu(x) = \mu(x^{**})$. Now we proceed to show that the kernel of θ_μ is μ. For any $x \in S$,

$$(\theta_\mu)_0(x) = \sup\{\mu(z) : x \wedge z^* = 0\} = \sup\{\mu(z) : x \le z^{**}\} \ge \mu(x) \tag{19}$$

Let $y \in S$ satisfying $x \le y^{**}$. Then $\mu(x) \ge \mu(y)$. This implies that $\mu(x)$ is an upper bound of $\{\mu(z) : x \le z^{**}\}$. This shows that $\mu(x) \ge (\theta_\mu)_0(x)$. Thus $\mu = (\theta_\mu)_0$. So μ is a kernel of θ_μ. □

Corollary 24. *μ is a kernel fuzzy ideal if and only if*

(1) $\mu(x) = \mu(x^{**})$,

(2) $\mu(k) = \sup\{\mu(x) \wedge \mu(y) : x^* \wedge y^* = k^*\}$.

Proof. Let μ be a kernel fuzzy ideal of S. Then $\mu((x^* \wedge y^*)^*) \ge \mu(x) \wedge \mu(y)$ for all $x, y \in S$.

(1) Since $x \le x^{**}$ and μ is a fuzzy ideal, we get $\mu(x) = \mu(x \wedge x^{**}) \ge \mu(x^{**})$. Again if $x = y$, then $\mu(x) \le \mu(x^{**})$. Thus $\mu(x^{**}) = \mu(x)$ for all $x \in S$.

(2) Let $x, y, k \in S$ such that $x^* \wedge y^* = k^*$. Then $(x^* \wedge y^*)^* = k^{**}$. By (1) and by the assumption, we have $\mu(k) \ge \mu(x) \wedge \mu(y)$. This implies $\mu(k) \ge \sup\{\mu(x) \wedge \mu(y) : x^* \wedge y^* = k^*,\ x, y, k \in S\}$. On the other hand, since $k^* = k^* \wedge k^*$ and $\mu(k) \le \mu(k)$, we have $\mu(k) \le \sup\{\mu(x) \wedge \mu(y) : x^* \wedge y^* = k^*,\ x, y, k \in S\}$. Thus $\mu(k) = \sup\{\mu(x) \wedge \mu(y) : x^* \wedge y^* = k^*,\ x, y, k \in S\}$.

Conversely, if (1) and (2) hold, for any $x, y \in S$ there exists $k \in S$ such that $x^* \wedge y^* = k^*$, then $\mu(k) \ge \mu(x) \wedge \mu(y)$. Thus by (1), we get that $\mu((x^* \wedge y^*)^*) = \mu(k^{**}) \ge \mu(x) \wedge \mu(y)$. So μ is a kernel fuzzy ideal of S. □

The set $S(S) = \{x^{**} : x \in S\}$ is called the skeleton of S. The elements of $S(S)$ are called skeletal.

Corollary 25. *Let $\alpha \in [0, 1]$. α_x is a kernel fuzzy ideal if and only if x is a skeletal element of S.*

Proof. Let α_x be a kernel fuzzy ideal. Then by Corollary 24(1) $\alpha_x(x^{**}) = \alpha_x(x)$, which implies $x^{**} \le x$. Since $x \le x^{**}$ for all $x \in S$, we get that $x = x^{**}$. Thus x is a skeletal element.

Conversely, Suppose that x is a skeletal element of S. To show α_x is a kernel fuzzy ideal, let $y, z \in S$ such that $y \le x$ and $z \le x$. Then $x^* \le y^* \wedge z^*$ and $\alpha_x(y) \wedge \alpha_x(z) = 1$. Thus $(y^* \wedge z^*)^* \le x^{**} = x$. So $\alpha_x((y^* \wedge z^*)^*) = 1$. This shows that $\alpha_x((y^* \wedge z^*)^*) \ge \alpha_x(y) \wedge \alpha_x(z)$. If $y \nleq x$ and $z \nleq x$, then trivially holds. Hence α_x is a kernel fuzzy ideal. □

Corollary 26. *The following conditions on S are equivalent:*

(1) *Every fuzzy ideal of S is a kernel fuzzy ideal.*

(2) *Every level principal fuzzy ideal is a kernel fuzzy ideal.*

(3) *S is a Boolean algebra.*

Theorem 27. *A fuzzy subset μ of S is a kernel fuzzy ideal if and only if each level subset of μ is kernel ideal of S.*

Proof. Let μ be a kernel fuzzy ideal of S. Then by Theorem 16 μ_t is an ideal of $S, \forall t \in [0, 1]$. To show μ_t is a kernel ideal, let $x, y \in \mu_t$. Then by the assumption, $\mu((x^* \wedge y^*)^*) \ge \mu(x) \wedge \mu(y) \ge t$. Thus $(x^* \wedge y^*)^* \in \mu_t$.

Conversely, suppose that every level subset of μ is a kernel ideal of S. Then by Theorem 16 μ is a fuzzy ideal of S. To show μ is a kernel fuzzy ideal, let $x, y \in S$ such that $\mu(x) = \alpha$ and $\mu(y) = \beta$. Then either $\alpha \le \beta$ or $\beta \le \alpha$. Without loss of generality, $\alpha \le \beta$. Then $\mu_\beta \subseteq \mu_\alpha$ and by the assumption, $(x^* \wedge y^*)^* \in \mu_\alpha$. This shows that $\mu((x^* \wedge y^*)^*) \ge \alpha = \alpha \wedge \beta$. Thus $\mu((x^* \wedge y^*)^*) \ge \mu(x) \wedge \mu(y)$. So μ is a kernel fuzzy ideal of S. □

Corollary 28. *A nonempty subset I of S is a kernel ideal of S if and only if χ_I is a kernel fuzzy ideal.*

Definition 29. A fuzzy ideal μ of S is called a $*$-fuzzy ideal if $\mu(x) = \mu(x^{**})$ for all $x \in S$.

Corollary 30. *Every kernel fuzzy ideal is a $*$-fuzzy ideal.*

Corollary 31. *If L is a pseudocomplemented distributive lattice, then a fuzzy ideal μ of L is a kernel fuzzy ideal if and only if it is a $*$-fuzzy ideal.*

Proof. Suppose μ is a kernel fuzzy ideal of L. Then by Corollary 24(1) μ is a $*$-fuzzy ideal of L.

Conversely, suppose μ is a $*$-fuzzy ideal. Since L is a pseudocomplemented distributive lattice, for any $x, y \in L$ we have $(x \vee y)^* = x^* \wedge y^*$. Now $\mu((x^* \wedge y^*)^*) = \mu((x \vee y)^{**})$. Since μ is a $*$-fuzzy ideal, we get $\mu((x^* \wedge y^*)^*) = \mu(x \vee y) \ge \mu(x) \wedge \mu(y)$. Thus μ is a kernel fuzzy ideal of L. □

Theorem 32. *A $*$-fuzzy ideal μ is a kernel fuzzy ideal if and only if $\mu(\sup_{S(S)}\{x^{**}, y^{**}\}) \ge \mu(x) \wedge \mu(y)$ for all $x, y \in S$.*

Proof. Let $x, y \in S$. Define $\sup_{S(S)}\{x^{**}, y^{**}\} = (x^* \wedge y^*)^*$. Since $z^{***} = z^*$ for all $z \in S$, we have $(x^* \wedge y^*)^* = (x^* \wedge y^*)^{***}$. Thus $(x^* \wedge y^*)^* \in S(S)$. Since μ is a kernel fuzzy ideal, $\mu(\sup_{S(S)}\{x^{**}, y^{**}\}) = \mu((x^* \wedge y^*)^*) \ge \mu(x) \wedge \mu(y)$. □

Theorem 33. *A fuzzy subset μ of S is a $*$-fuzzy ideal if and only if each level subset of μ is a $*$-ideal of S.*

Proof. Let μ be a $*$-fuzzy ideal of S. Then $\mu(x) = \mu(x^{**})$ for each $x \in S$ and by Theorem 16 μ_t is an ideal of $S, t \in [0, 1]$. To show μ_t is a $*$-ideal, let $x \in \mu_t$. Then $\mu(x^{**}) = \mu(x) \ge t$ and $x^{**} \in \mu_t$. Thus each level subset of μ is a $*$-fuzzy ideal.

Conversely, suppose every level fuzzy subset of μ is a $*$-ideal of S. Then by Theorem 16 μ is a fuzzy ideal of S. Since $x \le x^{**}$ and μ is a fuzzy ideal, we have $\mu(x^{**}) \le \mu(x)$. Let

$\mu(x) = t$. Then $x^{**} \in \mu_t$. Thus $\mu(x^{**}) \geq \mu(x)$. So μ is a $*$-fuzzy ideal of S. □

Corollary 34. *A nonempty subset I of S is a $*$-ideal of S if and only if χ_I is a $*$-ideal.*

Theorem 35. *Let μ be a kernel fuzzy ideal of S. Then the smallest $*$-fuzzy congruence on S with kernel μ is given by*

$$\theta_\mu(x, y) = \sup\{\mu(z) : x \wedge z^* = y \wedge z^*,\ z \in S\}. \tag{20}$$

Proof. It is shown in the proof of Theorem 23 that θ_μ is a $*$-fuzzy congruence with kernel μ. Now we proceed to show that θ_μ is the smallest $*$-fuzzy congruence with kernel μ.

Let η be a $*$-fuzzy congruence with kernel μ. For $x, y \in S$,

$$\theta_\mu(x, y) = \sup\{\mu(z) : x \wedge z^* = y \wedge z^*,\ z \in S\}. \tag{21}$$

Since η is a $*$-fuzzy congruence, we have $\eta(x \wedge z^*, x) \geq \eta(x, x) \wedge \eta(z^*, 1) = \eta(z^*, 1) \geq \eta(z, 0) = \mu(z)$. Similarly, $\eta(y \wedge z^*, y) \geq \mu(z)$. If $x \wedge z^* = y \wedge z^*$, then

$$\begin{aligned} \eta(x, y) &\geq \eta(x, x \wedge z^*) \wedge \eta(x \wedge z^*, y) \\ &= \eta(x, x \wedge z^*) \wedge \eta(y \wedge z^*, y) \geq \mu(z) \end{aligned} \tag{22}$$

This shows that $\eta(x, y)$ is an upper bound of $\{\mu(z) : x \wedge z^* = y \wedge z^*,\ z \in S\}$. Thus $\theta_\mu \subseteq \eta$. So θ_μ is the smallest $*$-fuzzy congruence with kernel μ. □

Lemma 36. *If μ and θ are $*$-fuzzy ideals of S, then so is $\mu \cup \theta$.*

Proof. Suppose μ and θ are $*$-fuzzy ideals of S. Clearly $(\mu \cup \theta)(0) = 1$. Let $x, y \in S$. Then $(\mu\cup\theta)(x\wedge y) = \mu(x\wedge y)\vee\theta(x\wedge y)$. Since μ and θ are fuzzy ideals, we have $(\mu\cup\theta)(x\wedge y) \geq (\mu(x)\vee\theta(x))\vee(\mu(y)\vee\theta(y)) = (\mu\cup\theta)(x)\vee(\mu\cup\theta)(y)$. Thus $\mu\cup\theta$ is a fuzzy ideal of S. Now $(\mu \cup \theta)(x) = \mu(x) \vee \theta(x)$. Since μ and θ are $*$-fuzzy ideals, we get that $(\mu\cup\theta)(x) = \mu(x^{**})\vee\theta(x^{**}) = (\mu\cup\theta)(x^{**})$ for all $x \in S$. Thus $\mu\cup\theta$ is a $*$-fuzzy ideal of S. □

The class of all $*$-fuzzy ideals of S is denoted by $FI^*(S)$. It is clear that the set $FI^*(S)$ of $*$-fuzzy ideals of S, ordered by fuzzy set inclusion, is a complete distributive lattice in which the lattice operations are fuzzy set-theoretic.

The class of all kernel fuzzy ideals of S is denoted by $FKI(S)$. We now prove that $FKI(S)$ is a complete distributive lattice.

Theorem 37. *If $\mu, \theta \in FKI(S)$, the supremum of μ and θ is given by*

$$\begin{aligned} &(\mu\underline{\vee}\theta)(x) \\ &\quad = \sup\left\{\mu(z) \wedge \theta(w) : x \leq (z^* \wedge w^*)^*,\ z, w \in S\right\}. \end{aligned} \tag{23}$$

Proof. Let $\eta = \mu\underline{\vee}\theta$. First, we need to show that η is a kernel fuzzy ideal of S. Since $0 \leq (0^* \wedge 0^*)^*$, we have $\eta(0) = 1$. For any $x, y \in S$,

$$\begin{aligned} \eta(x) &= \sup\left\{\mu(z) \wedge \theta(w) : x \leq (z^* \wedge w^*)^*,\ z, w \right. \\ &\quad \left. \in S\right\} \leq \sup\left\{\mu(z) \wedge \theta(w) : x \wedge y \right. \\ &\quad \left. \leq (z^* \wedge w^*)^*,\ z, w \in S\right\} = \eta(x \wedge y) \end{aligned} \tag{24}$$

This shows that $\eta(x \wedge y) \geq \eta(x) \vee \eta(y)$. Thus η is a fuzzy ideal of S.

We now show that η is a kernel fuzzy ideal. For any $x, y \in S$,

$$\begin{aligned} \eta(x) \wedge \eta(y) &= \sup\left\{\mu(z_1) \wedge \theta(w_1) : x \right. \\ &\quad \left. \leq (z_1^* \wedge w_1^*)^*,\ z_1, w_1 \in S\right\} \wedge \sup\left\{\mu(z_2) \right. \\ &\quad \left. \wedge \theta(w_2) : y \leq (z_2^* \wedge w_2^*)^*,\ z_2, w_2 \in S\right\} \\ &= \sup\left\{(\mu(z_1) \wedge \mu(z_2)) \wedge (\theta(w_1) \wedge \theta(w_2)) : x \right. \\ &\quad \left. \leq (z_1^* \wedge w_1^*)^*, y \leq (z_2^* \wedge w_2^*)^*\right\} \end{aligned} \tag{25}$$

If $x \leq (z_1^* \wedge w_1^*)^*$ and $y \leq (z_2^* \wedge w_2^*)^*$, then $z_1^* \wedge w_1^* \leq x^*$ and $z_2^* \wedge w_2^* \leq y^*$. Thus $(x^* \wedge y^*)^* \leq ((z_1^* \wedge z_2^*)^{**} \wedge (w_1^* \wedge w_2^*)^{**})^*$. Since μ and θ are kernel fuzzy ideals, we get that $\mu(z_1) \wedge \mu(z_2) \leq \mu((z_1^* \wedge z_2^*)^*)$ and $\theta(w_1) \wedge \theta(w_2) \leq \theta((w_1^* \wedge w_2^*)^*)$. Based on this we have

$$\begin{aligned} \eta(x) \wedge \eta(y) &\leq \sup\left\{\mu\left((z_1^* \wedge z_2^*)^*\right) \right. \\ &\quad \wedge \theta\left((w_1^* \wedge w_2^*)^*\right) : (x^* \wedge y^*)^* \\ &\quad \left. \leq \left((z_1^* \wedge z_2^*)^{**} \wedge (w_1^* \wedge w_2^*)^{**}\right)^*\right\} \leq \sup\left\{\mu(u) \right. \\ &\quad \left. \wedge \theta(v) : (x^* \wedge y^*)^* \leq (u^* \wedge v^*)^*,\ u, v \in S\right\} \\ &= \eta\left((x^* \wedge y^*)^*\right) \end{aligned} \tag{26}$$

Thus η is a kernel fuzzy ideal of S.

To show η is the smallest kernel fuzzy ideal of S, let λ be a kernel fuzzy ideal of S containing μ and θ. Then for any $x \in S$,

$$\begin{aligned} \eta(x) &= \sup\left\{\mu(z) \wedge \theta(w) : x \leq (z^* \wedge w^*)^*\right\} \\ &\leq \sup\left\{\lambda(z) \wedge \lambda(w) : x \leq (z^* \wedge w^*)^*\right\} \end{aligned} \tag{27}$$

If $x \leq (z^* \wedge w^*)^*$, then $\lambda(x) \geq \lambda(z) \wedge \lambda(w)$. This implies $\lambda(x)$ is an upper bound of $\{\lambda(z) \wedge \lambda(w) : x \leq (z^* \wedge w^*)^*\}$. Thus $\eta \subseteq \lambda$. Hence η is the smallest kernel fuzzy ideal containing μ and θ. □

Theorem 38. *The set $FKI(S)$ forms a complete distributive lattice with respect to inclusion ordering of fuzzy sets.*

Proof. Clearly $(FKI(S), \subseteq)$ is a partially ordered set. For $\mu, \theta \in FKI(S)$, clearly $\mu \wedge \theta, \mu\underline{\vee}\theta \in (FKI(S))$. So $(FKI(S), \wedge, \underline{\vee})$ is a lattice.

For $\mu, \theta, \eta \in FKI(S)$, clearly $(\mu \cap \theta)\underline{\vee}(\mu \cap \eta) \subseteq \mu \cap (\theta\underline{\vee}\eta)$. Now for any $x \in L$,

$$\begin{aligned} (\mu \cap (\theta\underline{\vee}\eta))(x) &= \sup\left\{\mu(x) \wedge (\theta(y) \wedge \eta(z)) : x \right. \\ &\quad \left. \leq (y^* \wedge z^*)^*,\ y, z \in S\right\} = \sup\left\{(\mu(x) \wedge \theta(y))\right. \end{aligned}$$

$$\begin{aligned}&\wedge(\mu(x)\wedge\eta(z)) : x \le (y^* \wedge z^*)^*\big\} \\ &= \sup\big\{(\mu(x)\wedge\theta(y^{**}))\wedge(\mu(x)\wedge\eta(z^{**})) : x \\ &\le (y^*\wedge z^*)^*\big\} \le \sup\big\{(\mu(x\wedge y^{**})\wedge\theta(x\wedge y^{**})) \\ &\wedge(\mu(x\wedge z^{**})\wedge\eta(x\wedge z^{**})) : x\le(y^*\wedge z^*)^*\big\} \\ &= \sup\big\{(\mu\cap\theta)(x\wedge y^{**})\wedge(\mu\cap\eta)(x\wedge z^{**}) : x \\ &\le (y^*\wedge z^*)^*\big\}\end{aligned} \tag{28}$$

If $x \le (y^* \wedge z^*)^*$, then $x \wedge y^* \wedge z^* = 0$. Since S is a pseudocomplemented semilattice, we have $y \wedge z^* = y \wedge (y \wedge z)^*$ for all $y, z \in S$. Now we proceed to show that $x \le ((x \wedge y^{**})^* \wedge (x \wedge z^{**})^*)^*$.

$$\begin{aligned}&x\wedge\left((x\wedge y^{**})^*\wedge(x\wedge z^{**})^*\right)^* \\ &= x\wedge\left(x\wedge\left((x\wedge y^{**})^*\wedge(x\wedge z^{**})^*\right)\right)^* \\ &= x\wedge\left(x\wedge(x\wedge y^{**})^*\wedge x\wedge(x\wedge z^{**})^*\right)^* \\ &= x\wedge(x\wedge y^*\wedge x\wedge z^*)^* = x\wedge(x\wedge y^*\wedge z^*)^* \\ &= x\wedge 0^* = x\end{aligned} \tag{29}$$

Thus $x \le ((x \wedge y^{**})^* \wedge (x \wedge z^{**})^*)^*$. Based on this fact we have

$$\begin{aligned}&(\mu\cap(\theta\underline{\vee}\eta))(x) \le \sup\big\{(\mu\cap\theta)(x\wedge y^{**}) \\ &\wedge(\mu\cap\eta)(x\wedge z^{**}) : x \\ &\le \left((x\wedge y^{**})^*\wedge(x\wedge z^{**})^*\right)^*\big\} \\ &\le \sup\big\{(\mu\cap\theta)(w)\wedge(\mu\cap\eta)(u) : x \\ &\le (w^*\wedge u^*)^*,\ w,u\in S\big\} \le ((\mu\cap\theta)\underline{\vee}(\mu\cap\eta))(x)\end{aligned} \tag{30}$$

Thus $\mu \cap (\theta\underline{\vee}\eta) = (\mu\cap\theta)\underline{\vee}(\mu\cap\eta)$. So $FKI(S)$ is distributive.

Now we show the completeness. Since $(FKI(S), \subseteq)$ is a poset and χ_S is greatest element of $FKI(S)$, it is enough to show that every subfamily of $FKI(S)$ has infimum. Let $\{\mu_\alpha : \alpha \in J\}$ be a subfamily of $FKI(S)$. Then $\bigcap_{\alpha\in J}\mu_\alpha$ is a fuzzy ideal of S. For any $x, y \in S$,

$$\begin{aligned}&\left(\bigcap_{\alpha\in J}\mu_\alpha\right)(x)\wedge\left(\bigcap_{\alpha\in J}\mu_\alpha\right)(y) \\ &= \inf\{\mu_\alpha(x):\alpha\in J\}\wedge\inf\{\mu_\beta(y):\beta\in J\} \\ &\le \inf\{\mu_\alpha(x)\wedge\mu_\alpha(y):\alpha\in J\} \\ &\le \inf\{\mu_\alpha((x^*\wedge y^*)^*):\alpha\in J\} \\ &= \left(\bigcap_{\alpha\in J}\mu_\alpha\right)((x^*\wedge y^*)^*)\end{aligned} \tag{31}$$

Thus $\bigcap_{\alpha\in I}\mu_\alpha$ is a kernel fuzzy ideal of L. So $FKI(S)$ is a complete distributive lattice. □

Corollary 39. *If L is a pseudocomplemented lattice, then $FI^*(L)$ forms a distributive lattice with respect to inclusion ordering of fuzzy sets.*

Proof. Clearly $(FI^*(L), \subseteq)$ is a partially ordered set. For $\mu, \theta \in FI^*(L)$, define

$$\begin{aligned}&\mu\wedge\theta = \mu\cap\theta, \\ &(\mu\underline{\vee}\theta)(x) \\ &\quad = \sup\{\mu(z)\wedge\theta(w) : x\le(z^*\wedge w^*)^*,\ z,w\in L\}.\end{aligned} \tag{32}$$

Obviously $\mu\wedge\theta, \mu\underline{\vee}\theta \in (FI^*(L))$. Let $\eta = \mu\underline{\vee}\theta$ and $x, y \in L$. Then

$$\begin{aligned}&\eta(x)\wedge\eta(y) = \sup\{\mu(z_1)\wedge\theta(w_1) : x \\ &\quad\le(z_1^*\wedge w_1^*)^*,\ z_1,w_1\in L\}\wedge\sup\{\mu(z_2) \\ &\quad\wedge\theta(w_2) : y\le(z_2^*\wedge w_2^*)^*,\ z_2,w_2\in L\} \\ &\quad= \sup\{(\mu(z_1)\wedge\mu(z_2))\wedge(\theta(w_1)\wedge\theta(w_2)) : x \\ &\quad\le(z_1^*\wedge w_1^*)^*, y\le(z_2^*\wedge w_2^*)^*\} = \sup\{\mu(z_1\vee z_2) \\ &\quad\wedge\theta(w_1\vee w_2) : x\le(z_1^*\wedge w_1^*)^*, y\le(z_2^*\wedge w_2^*)^*\}\end{aligned} \tag{33}$$

If $x \le (z_1^*\wedge w_1^*)^*$ and $y \le (z_2^*\wedge w_2^*)^*$, then $x\vee y \le (z_1^*\wedge w_1^*)^*\vee(z_2^*\wedge w_2^*)^*$, which implies $x\vee y \le ((z_1^*\wedge w_1^*)^*\vee(z_2^*\wedge w_2^*)^*)^{**} = ((z_1^*\wedge w_1^*)\wedge(z_2^*\wedge w_2^*))^* = ((z_1\vee z_2)^*\wedge(w_1\vee w_2)^*))^*$. Using this fact we have

$$\begin{aligned}&\eta(x)\wedge\eta(y)\le\sup\big\{\mu(z_1\vee z_2)\wedge\theta(w_1\vee w_2) : x \\ &\quad\vee y\le\left((z_1\vee z_2)^*\wedge(w_1\vee w_2)^*)\right)^*\big\} \\ &\quad\le\sup\{\mu(u_1)\wedge\theta(u_2) : x\vee y\le(u_1^*\wedge u_2^*)^*\} \\ &\quad= \eta(x\vee y)\end{aligned} \tag{34}$$

Thus $\eta(x\vee y) \ge \eta(x)\wedge\eta(y)$. So $FI^*(L)$ is a distributive lattice. □

Theorem 40. *If L is a pseudocomplemented distributive lattice, then the following statements are equivalent:*

(1) *if μ, θ are kernel fuzzy ideals of L, then so is $\mu\vee\theta$,*

(2) *L is a Stone lattice.*

Proof. (1) $\Longrightarrow$ (2): Let $x \in L$. Then x^* and x^{**} are skeletal elements of L. Thus by Corollary 25, α_{x^*} and $\alpha_{x^{**}}$ are kernel fuzzy ideals. By the assumption, $\alpha_{x^*}\vee\alpha_{x^{**}} = \alpha_{x^*\vee x^{**}}$ is a kernel fuzzy ideal. To prove our claim, it suffices to show that $x^*\vee x^{**} = 1$ for all $x \in L$. Since $\alpha_{x^*\vee x^{**}}$ is a kernel fuzzy ideal, then by Corollary 25, $x^*\vee x^{**} \in S(L)$. This implies $x^*\vee x^{**} = (x^*\vee x^{**})^{**} = (x^{**}\wedge x^*)^* = 1$ for all $x \in L$. Thus L is a Stone lattice.

(2) $\Longrightarrow$ (1): If L is a Stone lattice, then $(x \wedge y)^* = x^* \vee y^*$ for all $x, y \in L$. Let μ and θ be kernel fuzzy ideals. Then $\mu \vee \theta$ is a fuzzy ideal of L. To show $\mu \vee \theta$ is a kernel fuzzy ideal, it suffices to show that $(\mu \vee \theta)(x) = (\mu \vee \theta)(x^{**})$. Since $x \leq x^{**}$ and $\mu \vee \theta$ is a fuzzy ideal, we have $(\mu \vee \theta)(x^{**}) \leq (\mu \vee \theta)(x)$.

Now, $(\mu \vee \theta)(x) = \sup\{\mu(y) \wedge \theta(z) : x = y \vee z\}$. Since L is a Stone lattice, $x = y \vee z \Longrightarrow x^{**} = y^{**} \vee z^{**}$. Based on this we have

$$\begin{aligned}(\mu \vee \theta)(x) &\leq \sup\{\mu(y) \wedge \theta(z) : x^{**} = y^{**} \vee z^{**}\} \\ &= \sup\{\mu(y^{**}) \wedge \theta(z^{**}) : x^{**} = y^{**} \vee z^{**}\} \\ &= \sup\{\mu(w) \wedge \theta(u) : x^{**} = w \vee u\} \\ &= (\mu \vee \theta)(x^{**})\end{aligned} \tag{35}$$

Thus $\mu \vee \theta$ is a $*$-fuzzy ideal of L. So by Corollary 31, $\mu \vee \theta$ is a kernel fuzzy ideal of L □

If S and M are pseudocomplemented semilattices, then a semilattice morphism $f : S \longrightarrow M$ will be called a $*$-morphism if $f(x^*) = (f(x))^*$ for all $x \in S$.

Lemma 41. *Let $f : S \longrightarrow M$ be an epimorphism. Then*

(1) *If μ is a fuzzy ideal of S, then $f(\mu)$ is a fuzzy ideal of M.*

(2) *If θ is a fuzzy ideal of M, then $f^{-1}(\theta)$ is a fuzzy ideal of S.*

Proof. (1) If μ is a fuzzy ideal of S, then $f(\mu)(0_M) = \sup\{\mu(a) : a \in f^{-1}(0_M)\} \geq \mu(0_S) = 1$. Thus $f(\mu)(0_M) = 1$. For any $x, y \in M$,

$$\begin{aligned}f(\mu)(x) &= \sup\{\mu(z) : z \in f^{-1}(x)\} \\ &\leq \sup\{\mu(z) \vee \mu(w) : z \in f^{-1}(x),\ w \in f^{-1}(y)\}\end{aligned} \tag{36}$$

Since $z \in f^{-1}(x)$ and $w \in f^{-1}(y)$, we get that $z \wedge w \in f^{-1}(x \wedge y)$. Based on this we have

$$\begin{aligned}f(\mu)(x) &\leq \sup\{\mu(z \wedge w) : z \wedge w \in f^{-1}(x \wedge y)\} \\ &\leq \sup\{\mu(u) : u \in f^{-1}(x \wedge y)\} \\ &= f(\mu)(x \wedge y)\end{aligned} \tag{37}$$

Similarly, $f(\mu)(y) \leq f(\mu)(x \wedge y)$. Thus $f(\mu)(x \wedge y) \geq f(\mu)(x) \vee f(\mu)(y)$. So $f(\mu)$ is a fuzzy ideal of M.

(2) If θ is a fuzzy ideal of M, then $f^{-1}(\theta)(0_S) = \theta(f(0_S)) = \theta(0_M) = 1$. For any $x, y \in S$, we have $f^{-1}(\theta)(x \wedge y) = (\theta)(f(x) \wedge f(y)) \geq (\theta)(f(x)) \vee (\theta)(f(y)) = f^{-1}(\theta)(x) \vee f^{-1}(\theta)(y)$. Thus $f^{-1}(\theta)$ is a fuzzy ideal of S. □

Theorem 42. *Let $f : S \longrightarrow M$ be a $*$-epimorphism. Then*

(1) *If μ is a kernel fuzzy ideal of S, then $f(\mu)$ is a kernel fuzzy ideal of M.*

(2) *If θ is a kernel fuzzy ideal of M, then $f^{-1}(\theta)$ is a kernel fuzzy ideal of S.*

Proof. (1) Let μ be a kernel fuzzy ideal of S. Then by Lemma 41(1) $f(\mu)$ is a fuzzy ideal of M. To show $f(\mu)$ is a kernel fuzzy ideal, let $x, y \in M$. Then

$$\begin{aligned}&f(\mu)(x) \wedge f(\mu)(y) \\ &\quad = \sup\{\mu(z) : z \in f^{-1}(x),\ z \in S\} \\ &\qquad \wedge \sup\{\mu(w) : w \in f^{-1}(y),\ w \in S\} \\ &\quad = \sup\{\mu(z) \wedge \mu(w) : z \in f^{-1}(x),\ w \in f^{-1}(y)\}\end{aligned} \tag{38}$$

If $z \in f^{-1}(x)$ and $w \in f^{-1}(y)$, then $(z^* \wedge w^*)^* \in f^{-1}((x^* \wedge y^*)^*)$. Since μ is a kernel fuzzy ideal of S, we have that $\mu(z) \wedge \mu(w) \leq \mu((z^* \wedge w^*)^*)$. Based on this fact we have

$$\begin{aligned}&f(\mu)(x) \wedge f(\mu)(y) \\ &\quad \leq \sup\{\mu((z^* \wedge w^*)^*) : (z^* \wedge w^*)^* \\ &\quad \in f^{-1}((x^* \wedge y^*)^*)\} \leq \sup\{\mu(u) : u \\ &\quad \in f^{-1}((x^* \wedge y^*)^*),\ u \in S\} = f(\mu)((x^* \wedge y^*)^*)\end{aligned} \tag{39}$$

This shows that $f(\mu)((x^* \wedge y^*)^*) \geq f(\mu)(x) \wedge f(\mu)(y)$ for any $x, y \in M$. Hence $f(\mu)$ is a kernel fuzzy ideal of M.

(2) Let θ be a kernel fuzzy ideal of M. Then by Lemma 41(2) $f^{-1}(\theta)$ is a fuzzy ideal of S. To show $f^{-1}(\theta)$ is a kernel fuzzy ideal, let $x, y \in S$. Then

$$\begin{aligned}f^{-1}(\theta)((x^* \wedge y^*)^*) &= \theta(f((x^* \wedge y^*)^*)) \\ &= \theta((f(x)^* \wedge f(y)^*)^*) \\ &\geq f^{-1}(\theta)(x) \wedge f^{-1}(\theta)(y).\end{aligned} \tag{40}$$

Thus $f^{-1}(\theta)$ is a kernel fuzzy ideal of S. □

Theorem 43. *Let $f : S \longrightarrow M$ be a $*$-epimorphism. Then the map $g : FKI(S) \longrightarrow FKI(M)$ defined by $\mu \longmapsto f(\mu)$ is a lattice epimorphism.*

Proof. Clearly $\chi_{\{0_S\}}, \chi_S \in FKI(S)$. Since $\mathrm{Ker} f = \{0\}$ and f is onto, we get $f(\chi_{\{0_S\}}) = \chi_{\{0_M\}}$ and $f(\chi_S) = \chi_M$. This implies $g(\chi_{\{0_S\}}) = \chi_{\{0_M\}}$ and $g(\chi_S) = \chi_M$.

Let $\mu, \theta \in FKI(S)$. Then $\mu \cap \theta$ and $\mu \underline{\vee} \theta$ are kernel fuzzy ideals of S. Thus $f(\mu \cap \theta)$ and $f(\mu \underline{\vee} \theta)$ are kernel fuzzy ideals of M. Since $\mu \cap \theta \subseteq \mu$ and $\mu \cap \theta \subseteq \theta$, we have $f(\mu \cap \theta) \subseteq f(\theta) \cap f(\mu)$. For any $y \in M$,

$$\begin{aligned}&(f(\mu) \cap f(\theta))(y) \\ &\quad = \sup\{\mu(z) : z \in f^{-1}(y),\ z \in S\} \\ &\qquad \wedge \sup\{\theta(w) : w \in f^{-1}(y),\ w \in S\}.\end{aligned} \tag{41}$$

Since f is a homomorphism and $f(z) = y$, $f(w) = y$, we get $f(z \wedge w) = y$. Using this fact, we have

$$\begin{aligned}
&(f(\mu) \cap f(\theta))(y) \\
&\quad \leq \sup\{\mu(z \wedge w) : z \wedge w \in f^{-1}(y)\} \\
&\qquad \wedge \sup\{\theta(z \wedge w) : z \wedge w \in f^{-1}(y)\} \\
&\quad = \sup\{\mu(z \wedge w) \wedge \theta(z \wedge w) : z \wedge w \in f^{-1}(y)\} \\
&\quad = \sup\{(\mu \cap \theta)(z \wedge w) : z \wedge w \in f^{-1}(y)\} \\
&\quad \leq \sup\{(\mu \cap \theta)(u) : u \in f^{-1}(y)\} \\
&\quad = f(\mu \cap \theta)(y).
\end{aligned} \tag{42}$$

So $f(\mu) \cap f(\theta) = f(\mu \cap \theta)$. Again clearly, $f(\mu) \underline{\vee} f(\theta) \subseteq f(\mu \underline{\vee} \theta)$. For any $x \in M$,

$$\begin{aligned}
&(f(\mu) \underline{\vee} f(\theta))(x) = \sup\{f(\mu)(x_1) \wedge f(\theta)(x_2) : x \\
&\quad \leq (x_1^* \wedge x_2^*)^*\} \\
&\quad = \sup\{\sup\{\mu(w_1) : w_1 \in f^{-1}(x_1)\} \\
&\quad \wedge \sup\{\mu(w_2) : w_2 \in f^{-1}(x_2)\} : x \leq (x_1^* \wedge x_2^*)^*\}
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
&f(\mu \underline{\vee} \theta)(x) = \sup\{(\mu \underline{\vee} \theta)(z) : z \in f^{-1}(x),\ z \in S\} \\
&\quad = \sup\{\sup\{\mu(z_1) \wedge \theta(z_2) : z \leq (z_1^* \wedge z_2^*)^*\} : z \\
&\quad \in f^{-1}(x)\}.
\end{aligned} \tag{44}$$

If $f(z) = x$ and $z \leq (z_1^* \wedge z_2^*)^*$, then $x = f(z) \leq (f(z_1)^* \wedge f(z_2)^*)^*$. Put $f(z_1) = y_1$ and $f(z_2) = y_2$. Then $z_1 \in f^{-1}(y_1), z_2 \in f^{-1}(y_2)$ and $x \leq (y_1^* \wedge y_2^*)^*$. Based on this fact we have

$$\begin{aligned}
&f(\mu \underline{\vee} \theta)(x) \leq \sup\{\sup\{\mu(z_1) : z_1 \in f^{-1}(y_1)\} \\
&\quad \wedge \sup\{\mu(z_2) : z_2 \in f^{-1}(y_2)\} : x \leq (y_1^* \wedge y_2^*)^*\} \\
&\quad = \sup\{f(\mu)(y_1) \wedge f(\mu)(y_2) : x \leq (y_1^* \wedge y_2^*)^*\} \\
&\quad = (f(\mu) \underline{\vee} f(\theta))(x)
\end{aligned} \tag{45}$$

Thus $f(\mu \underline{\vee} \theta) = f(\mu) \underline{\vee} f(\theta)$. So g is a homomorphism. To show g is an epimorphism, let $\mu \in FKI(M)$. Then $f^{-1}(\mu) \in FKI(M)$. Since f is onto, we have $f(f^{-1}(\mu)) = \mu$. Thus g is onto. So g is a lattice epimorphism. □

4. ∗-Fuzzy Filters

Turning our attention to fuzzy filters, we first introduce the concept of ∗-fuzzy filter of a pseudocomplemented semilattice. We prove that, in contrast to the situation concerning fuzzy ideals, every fuzzy filter of a pseudocomplemented semilattice is the cokernel of a ∗-fuzzy congruence. Moreover, we have shown that there is an isomorphism between the class of kernel fuzzy ideals and the class of ∗-fuzzy filters of a pseudocomplemented semilattice.

Now we define a fuzzy filter of a semilattice.

Definition 44. A fuzzy subset μ of S is called a fuzzy filter of S if, for all $x, y \in S$, the following conditions are satisfied:

(1) $\mu(1) = 1$,

(2) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$,

(3) $\mu(x) \leq \mu(y)$ whenever $x \leq y$.

Theorem 45. *A fuzzy subset μ of S is a fuzzy filter of S if and only if*

$$\begin{aligned}
&\mu(1) = 1, \\
&\mu(x \wedge y) = \mu(x) \wedge \mu(y) \quad \text{for all } x, y \in S.
\end{aligned} \tag{46}$$

Proof. Let μ be a fuzzy filter of S and $x, y \in S$. Since $x \wedge y \leq x$ and $x \wedge y \leq y$, by the assumption we get that $\mu(x \wedge y) \leq \mu(x) \wedge \mu(y)$. Thus $\mu(1) = 1$ and $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$ for all $x, y \in S$.

Conversely, suppose the condition holds. Then conditions (1) and (2) of Definition 44 are satisfied. Let $x, y \in S$ such that $x \leq y$. Then $\mu(x) = \mu(x \wedge y) = \mu(x) \wedge \mu(y)$. Thus $\mu(x) \leq \mu(y)$. So μ is a fuzzy filter of S. □

Theorem 46. *A fuzzy subset μ of S is a fuzzy filter of S if and only if each level subset of μ is a filter of S. (In particular, a nonempty subset F of S is a filter of S if and only if χ_F is a fuzzy filter of S.)*

Proof. Suppose μ is a fuzzy filter of S. Clearly $1 \in \mu_t$ for all $t \in [0, 1]$. To show μ_t is a filter of S, let $x, y \in \mu_t$. Then $\mu(x) \wedge \mu(y) \geq t$. Since μ is a fuzzy filter of S, we get $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y) \geq t$. Thus $x \wedge y \in \mu_t$. Again, if $x \in \mu_t$ and $z \in S$ such that $x \leq z$, then by Theorem 45 we have $\mu(x) = \mu(x \wedge z) = \mu(x) \wedge \mu(z)$. Thus $\mu(z) \geq \mu(x) \geq t$. This implies $z \in \mu_t$. Hence μ_t is a filter of S for all $t \in [0, 1]$.

Suppose conversely that the condition holds. Then clearly $\mu(1) = 1$. Let $x, y \in S$ such that $\mu(x) = \alpha$ and $\mu(y) = \beta$. Then $x \in \mu_\alpha$ and $y \in \mu_\beta$. Since $\alpha, \beta \in [0, 1]$, either $\alpha \leq \beta$ or $\beta \leq \alpha$. Without loss of generality $\alpha \leq \beta$, then $\mu_\beta \subseteq \mu_\alpha$. Which implies $x, y \in \mu_\alpha$. Since μ_α is a filter, we get that $x \wedge y \in \mu_\alpha$. Thus $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$. Finally, let $x, y \in S$ such that $x \leq y$. If $\mu(x) = \alpha$, then $x \in \mu_\alpha$ and $x \leq y$. By the assumption, we have $y \in \mu_\alpha$. Thus $\mu(x) \leq \mu(y)$. So μ is a fuzzy filter of S. □

Theorem 47. *If θ is a fuzzy congruence on S, then the fuzzy congruence class θ_1 of S determined by θ and 1 is a fuzzy filter of S.*

Proof. Suppose θ be a fuzzy congruence on S. Then $\theta(1, 1) = 1$. Thus $\theta_1(1) = 1$. Let $x, y \in S$. Then $\theta_1(x) \wedge \theta_1(y) = \theta(x, 1) \wedge (y, 1)$. Since θ is a fuzzy congruence, we have $\theta_1(x) \wedge \theta_1(y) \leq \theta(x \wedge y, 1) = \theta_1(x \wedge y)$. Thus $\theta_1(x \wedge y) \geq \theta_1(x) \wedge \theta_1(y)$. Let $x \leq y$. Then $\theta_1(x) = \theta(x, 1) \wedge \theta(y, y) \leq \theta(x \wedge y, y) = \theta(x, y)$. Since θ is a fuzzy congruence, we have $\theta(y, 1) \geq \theta(y, x) \wedge$

$\theta(x,1) = \theta(x,1)$. Thus $\theta_1(x) \leq \theta_1(y)$. So θ_1 is a fuzzy filter of S. □

Definition 48. A fuzzy filter μ of S is called a cokernel fuzzy filter if $\mu = \theta_1$, where θ is a $*$-fuzzy congruence on S.

Theorem 49. *If η is a fuzzy filter of S, then the fuzzy relation θ_η defined by*

$$\theta_\eta(x,y) = \sup\{\eta(z) : x \wedge z = y \wedge z,\ z \in S\} \tag{47}$$

is a $*$-fuzzy congruence with cokernel fuzzy filter η; moreover, θ_η is the smallest $*$-fuzzy congruence.

Proof. It is clear that θ_η is a fuzzy congruence on S. Now we proceed to show that the fuzzy cokernel of θ_η is η. For any $x \in S$,

$$\begin{aligned}\theta_\mu(x,1) &= \sup\{\eta(z) : x \wedge z = z\}\\ &= \sup\{\eta(z) : z \leq x\} \geq \eta(x)\end{aligned} \tag{48}$$

Let $w \in S$ satisfying $w \leq x$. Then $\eta(x) \geq \eta(w)$. This implies that $\eta(x)$ is an upper bound of $\{\eta(z) : w \leq x\}$. Thus $\eta = (\theta_\mu)_1$. So η is a fuzzy cokernel of θ_η.

To show θ_η is a $*$-fuzzy congruence on S, let $x \in S$. Then

$$\begin{aligned}\theta_\eta(x^*,1) &= \sup\{\eta(z) : x^* \wedge z = z\}\\ &= \sup\{\eta(z) : z \leq x^*\},\\ \theta_\eta(x,0) &= \sup\{\eta(w) : x \wedge w = w\}\\ &= \sup\{\eta(w) : w \leq x^*\} \leq \theta_\eta(x^*,1).\end{aligned} \tag{49}$$

Thus θ_η is a $*$-fuzzy congruence on S. Finally, let λ be a $*$-fuzzy congruence with cokernel η and $x, y, z \in S$. Then $\eta(z) = \lambda(1,z) = \lambda(1,z)\wedge\lambda(x,x) \leq \lambda(x, x\wedge z)$. Similarly, $\lambda(y\wedge z, y) \geq \eta(z)$. Let $x\wedge z = y\wedge z$. Since λ is a fuzzy congruence, we have that

$$\begin{aligned}\lambda(x,y) &\geq \lambda(x, x\wedge z) \wedge \lambda(x\wedge z, y)\\ &= \lambda(x, x\wedge z) \wedge \lambda(y\wedge z, y) \geq \eta(z).\end{aligned} \tag{50}$$

This implies that $\lambda(x,y)$ is an upper bound of $\{\eta(z) : x \wedge z = y \wedge z\}$. Thus $\lambda(x,y) \geq \theta_\eta(x,y)$. So θ_η is the smallest $*$-fuzzy congruence with cokernel η. □

We now observe that the condition for fuzzy filters that is dual to that given in Theorem 23, namely, $\eta((x^* \wedge y^*)^*) \leq \eta(x) \wedge \eta(y)$, is of no interest; for if this held, then we would have $\eta(1) = \eta((x^* \wedge x^{**})^*) \leq \eta(x) \wedge \eta(x^*) = \eta(0)$. Thus $\eta = \chi_S$. However, the condition that is dual to condition (1) of Corollary 24 is of considerable interest.

Definition 50. A fuzzy filter μ of S is called a $*$-fuzzy filter if $\mu(x) = \mu(x^{**})$ for all $x \in L$.

Theorem 51. *A fuzzy subset μ of S is a $*$-fuzzy filter if and only if each level subset of μ is a $*$-filter of S.*

Proof. Suppose a fuzzy subset μ of S is a $*$-fuzzy filter. Then $\mu(x) = \mu(x^{**})$ for all $x \in S$ and by Theorem 46 every level subset of μ is a filter of S. To show μ_α is a $*$-filter, let $x^{**} \in \mu_\alpha, \alpha \in [0,1]$. Then by the assumption $x \in \mu_\alpha$. Thus μ_α is a $*$-filter of S.

Conversely, suppose that μ_α is a $*$-filter for all $\alpha \in [0,1]$. Then $x^{**} \in \mu_\alpha \Longrightarrow x \in \mu_\alpha$ and by Theorem 46, μ is a fuzzy filter of S. Since $x \leq x^{**}$ and μ is a fuzzy filter, then $\mu(x) \leq \mu(x^{**})$ for each $x \in S$. If $\mu(x^{**}) = \alpha$, then by the assumption $x \in \mu_\alpha$. This shows that $\mu(x) \geq \mu(x^{**})$. Thus $\mu(x) = \mu(x^{**})$ for all $x \in S$. So μ is a $*$-fuzzy filter of S. □

Corollary 52. *A nonempty subset F of S is a $*$-filter of S if and only if χ_F is a $*$-fuzzy filter.*

The class of fuzzy filters and $*$-fuzzy filters of S are denoted by $FF(S)$ and $FF^*(S)$ respectively.

Lemma 53. *Let μ be a fuzzy filter of S. Define $\alpha(\mu)(x) = \mu(x^*)$. Then $\alpha(\mu)$ is a kernel fuzzy ideal of S.*

Proof. To show $\alpha(\mu)$ is a fuzzy ideal of S, it is enough to show that $\alpha(\mu)(0) = 1$ and $\alpha(\mu)(x) \geq \alpha(\mu)(y)$ whenever $x \leq y$. Clearly $\alpha(\mu)(0) = 1$. If $x, y \in S$ such that $x \leq y$, then $y^* \leq x^*$. Since μ is a fuzzy filter, we get that $\mu(y^*) \leq \mu(x^*)$. This implies $\alpha(\mu)(x) \geq \alpha(\mu)(y)$. Thus $\alpha(\mu)$ is a fuzzy ideal of S. To show $\alpha(\mu)$ is a kernel fuzzy ideal, let $x, y \in S$. Then $\alpha(\mu)((x^* \wedge y^*)^*) = \mu(x^* \wedge y^*) \geq \mu(x^*) \wedge \mu(y^*) = \alpha(\mu)(x) \wedge \alpha(\mu)(y)$. Thus $\alpha(\mu)$ is a kernel fuzzy ideal of S. Now we can define a mapping $\alpha : FF(S) \longrightarrow FKI(S)$ by $\mu \longmapsto \alpha(\mu)$. □

Lemma 54. *Let θ be a kernel fuzzy ideal of S. Define $\beta(\theta)(x) = \theta(x^*)$. Then $\beta(\theta)$ is a $*$-fuzzy filter of S.*

Proof. Clearly $\beta(\theta)(1) = 1$. If $x, y \in S$ and $x \leq y$, then $y^* \leq x^*$ and $\beta(\theta)(x) \leq \beta(\theta)(y)$. We now show that $\beta(\theta)(x \wedge y) \geq \beta(\theta)(x) \wedge \beta(\theta)(y)$. Let $x, y \in S$. Then $(x \wedge y)^* = (x^{**} \wedge y^{**})^*$. Now $\beta(\theta)(x \wedge y) = \mu((x^{**} \wedge y^{**})^*) \geq \mu(x^*) \wedge \mu(y^*) = \beta(\theta)(x) \wedge \beta(\theta)(y)$. Thus $\beta(\theta)$ is a fuzzy ideal of S. Finally, for each $x \in S$, $\beta(\theta)(x) = \theta(x^*) = \beta(\theta)(x^{**})$. Hence $\beta(\theta)$ is a $*$-fuzzy ideal of S. Now we can define a mapping $\beta : FKI(S) \longrightarrow FF(S)$ by $\theta \longmapsto \beta(\theta)$. □

Theorem 55. *(1) For any fuzzy filter μ of S, we have $\mu \subseteq \beta(\alpha(\mu))$.*

(2) For every $\mu \in FKI(S)$, we have $\mu = \alpha(\beta(\mu))$.

Proof. (1) For any $x \in S$, we have $\beta(\alpha(\mu))(x) = \mu(x^{**})$. Since $x \leq x^{**}$ for each $x \in S$ and μ is a fuzzy filter, we have $\mu(x) \leq \mu(x^{**})$. Thus $\mu(x) \leq \beta(\alpha(\mu))(x)$ for all $x \in S$. So $\mu \subseteq \beta(\alpha(\mu))$.

(2) Let μ be a kernel fuzzy ideal of S. Then by Corollary 24(1) we have $\mu(x) = \mu(x^{**})$ for all $x \in S$. Thus $\mu = \alpha(\beta(\mu))$. □

Corollary 56. *$\beta(\alpha(\mu)) = \mu$ if and only if μ is a $*$-fuzzy filter.*

Lemma 57. *For any fuzzy filter μ of S, the map $\mu \longrightarrow \beta(\alpha(\mu))$ is a closure operator on $FF(S)$. That is,*

(1) $\mu \subseteq \beta(\alpha(\mu))$,

(2) $\beta\alpha(\beta(\alpha(\mu))) = \beta(\alpha(\mu))$,

(3) $\mu \subseteq \theta \Longrightarrow \beta(\alpha(\mu)) \subseteq \beta(\alpha(\theta))$.

Proof. (1) Since μ is a fuzzy filter of S, by Theorem 55(1) we have $\mu \subseteq \beta(\alpha(\mu))$.

(2) Since μ is a fuzzy filter, by Lemma 53 $\alpha(\mu)$ is a kernel fuzzy ideal of S. Again by Lemma 54, $\beta(\alpha(\mu))$ is a $*$-fuzzy filter. Thus by Corollary 56, $\beta\alpha(\beta(\alpha(\mu))) = \beta(\alpha(\mu))$.

(3) For any $x \in S$, we have $\beta(\alpha(\mu))(x) = \mu(x^{**})$ and $\beta(\alpha(\theta))(x) = \theta(x^{**})$. Since $\mu \subseteq \theta$, we get that $\mu(x^{**}) \leq \theta(x^{**})$. This shows that $\beta(\alpha(\mu))(x) \leq \beta(\alpha(\theta))(x)$ for all $x \in S$. Thus $\beta(\alpha(\mu)) \subseteq \beta(\alpha(\theta))$. □

The class of all $*$-fuzzy filters of S is denoted by $FF^*(S)$. We now prove that $FF^*(S)$ is a complete distributive lattice.

Theorem 58. *If $\mu, \theta \in FF^*(S)$, the supremum of μ and θ is given by*

$$(\mu \underline{\vee} \theta)(x) = \sup\{\mu(z) \wedge \theta(w) : x^* \leq (z \wedge w)^*,\ z, w \in S\}. \quad (51)$$

Proof. Let $\eta = \mu \underline{\vee} \theta$. We need to show that η is a $*$-fuzzy filter of S. Since $1^* \leq (1 \wedge 1)^*$, we have $\eta(1) = 1$. For any $x, y \in S$,

$$\begin{aligned}\eta(x) \wedge \eta(y) &= \sup\{\mu(z_1) \wedge \theta(w_1) : x^* \leq (z_1 \wedge w_1)^*,\ z_1, w_1 \in S\} \wedge \sup\{\mu(z_2) \wedge \theta(w_2) : y^* \leq (z_2 \wedge w_2)^*,\ z_2, w_2 \in S\} \\ &= \sup\{(\mu(z_1) \wedge \mu(z_2)) \wedge (\theta(w_1) \wedge \theta(w_2)) : x^* \leq (z_1 \wedge w_1)^*, y^* \leq (z_2 \wedge w_2)^*\}\end{aligned} \quad (52)$$

If $x^* \leq (z_1 \wedge w_1)^*$ and $y^* \leq (z_2 \wedge w_2)^*$, then $(x \wedge y)^{**} = x^{**} \wedge y^{**} \geq (z_1 \wedge w_1)^{**} \wedge (z_2 \wedge w_2)^{**} = ((z_1 \wedge z_2) \wedge (w_1 \wedge w_2))^{**}$. Thus $(x \wedge y)^* \leq ((z_1 \wedge z_2) \wedge (w_1 \wedge w_2))^*$. Using this fact we have

$$\eta(x) \wedge \eta(y) \leq \sup\{\mu(z_1 \wedge z_2) \wedge \theta(w_1 \wedge w_2) : (x \wedge y)^* \leq ((z_1 \wedge z_2) \wedge (w_1 \wedge w_2))^*\} \leq \sup\{\mu(u_1) \wedge \theta(u_2) : (x \wedge y)^* \leq (u_1 \wedge u_2)^*\} = \eta(x \wedge y) \quad (53)$$

Thus $\eta(x \wedge y) \geq \eta(x) \wedge \eta(y)$.

On the other hand, if $x \leq y$, then $y^* \leq x^*$ and $\eta(x) \leq \eta(y)$. Now we proceed to show that η is a $*$-fuzzy filter of S. For any $x \in S$, $\eta(x^{**}) = \sup\{\mu(z) \wedge \theta(w) : x^* \leq (z \wedge w)^*\} = \eta(x)$. Thus η is a $*$-fuzzy filter of S.

To show η is the smallest $*$-fuzzy filter containing μ and θ, let λ be a $*$-fuzzy filter containing μ and θ. Then for any $x \in S$,

$$\begin{aligned}\eta(x) &= \sup\{\mu(z) \wedge \theta(w) : x^* \leq (z \wedge w)^*\} \\ &\leq \sup\{\lambda(z) \wedge \lambda(w) : x^* \leq (z \wedge w)^*\}\end{aligned} \quad (54)$$

If $x^* \leq (z \wedge w)^*$, then $(z \wedge w)^{**} \leq x^{**}$. Since λ is a $*$-fuzzy filter, $\lambda(z \wedge w) \leq \lambda(x)$. Thus $\lambda(z) \wedge \lambda(w) \leq \lambda(x)$. So $\lambda(x)$ is an upper bound of $\{\lambda(z) \wedge \lambda(w) : x^* \leq (z \wedge w)^*\}$. Thus $\eta(x) \leq \lambda(x)$. So η is the smallest $*$-fuzzy filter containing μ and θ. □

Theorem 59. *The set $FF^*(S)$ forms a complete distributive lattice with respect to inclusion ordering of fuzzy sets.*

Proof. Clearly $(FF^*(S), \subseteq)$ is a partially ordered set. For $\mu, \theta \in FF^*(S)$, clearly $\mu \wedge \theta, \mu \underline{\vee} \theta \in FF^*(S)$. Thus $(FF^*(S), \wedge, \underline{\vee})$ is a lattice.

For $\mu, \theta, \eta \in FF^*(S)$, clearly $(\mu \cap \theta) \underline{\vee} (\mu \cap \eta) \subseteq \mu \cap (\theta \underline{\vee} \eta)$. For any $x \in S$,

$$\begin{aligned}(\mu \cap (\theta \underline{\vee} \eta))(x) &= \sup\{\mu(x) \wedge (\theta(y) \wedge \eta(z)) : x^* \leq (y \wedge z)^*,\ y, z \in S\} \\ &= \sup\{(\mu(x) \wedge \theta(y)) \wedge (\mu(x) \wedge \eta(z)) : x^* \leq (y \wedge z)^*\}\end{aligned} \quad (55)$$

If $x^* \leq (y \wedge z)^*$, then $x^* \wedge (y \wedge z)^{**} = 0$. Now we proceed to show that $x^* \leq ((x^* \wedge y^*)^* \wedge (x^* \wedge z^*)^*)^*$.

$$\begin{aligned}x^{**} &\wedge ((x^* \wedge y^*)^* \wedge (x^* \wedge z^*)^*) \\ &= ((x^* \wedge y^*)^* \wedge (x^* \wedge z^*)^*) \wedge (((x^* \wedge y^*)^* \wedge (x^* \wedge z^*)^*) \wedge x^*)^* \\ &\qquad \text{as } y \wedge z^* = y \wedge (y \wedge z)^* \\ &= ((x^* \wedge y^*)^* \wedge (x^* \wedge z^*)^*) \wedge (x^* \wedge (x^* \wedge y^*)^* \wedge x^* \wedge (x^* \wedge z^*)^*)^* \\ &= ((x^* \wedge y^*)^* \wedge (x^* \wedge z^*)^*) \wedge (x^* \wedge y^{**} \wedge x^* \wedge z^{**})^* \\ &= ((x^* \wedge y^*)^* \wedge (x^* \wedge z^*)^*) \wedge (x^* \wedge (y \wedge z)^{**})^* \\ &= ((x^* \wedge y^*)^* \wedge (x^* \wedge z^*)^*) \wedge 0^* \\ &= (x^* \wedge y^*)^* \wedge (x^* \wedge z^*)^*\end{aligned} \quad (56)$$

Thus $x^* \leq ((x^* \wedge y^*)^* \wedge (x^* \wedge z^*)^*)^*$. Since $x, y \leq (x^* \wedge y^*)^*$ and $x, z \leq (x^* \wedge z^*)^*$, we have that,

$$\begin{aligned}(\mu \cap (\theta \underline{\vee} \eta))(x) &\leq \sup\Big\{(\mu \cap \theta)\big((x^* \wedge y^*)^*\big) \\ &\wedge (\mu \cap \eta)\big((x^* \wedge z^*)^*\big) : x^* \\ &\leq \big((x^* \wedge y^*)^* \wedge (x^* \wedge z^*)^*\big)^*\Big\} \\ &\leq \sup\{(\mu \cap \theta)(w) \wedge (\mu \cap \eta)(u) : x^* \\ &\leq (w \wedge u)^*,\ u, w \in S\} \leq ((\mu \cap \theta) \underline{\vee} (\mu \cap \eta))(x)\end{aligned} \tag{57}$$

Thus $\mu \cap (\theta \underline{\vee} \eta) = (\mu \cap \theta) \underline{\vee} (\mu \cap \eta)$. So $FF^*(S)$ is distributive.

Let $\{\mu_\alpha : \alpha \in J\}$ be a subfamily of $FF^*(S)$. Then $\bigcap_{\alpha \in J} \mu_\alpha$ is a $*$-fuzzy filter of S. Thus $FF^*(S)$ is a complete distributive lattice. □

Corollary 60. *If L is a pseudocomplemented lattice, then $FF^*(L)$ forms a distributive lattice with respect to inclusion ordering of fuzzy sets.*

Theorem 61. $FF^*(S) \cong FKI(S)$.

Proof. Define $\widehat{\beta} : FKI(S) \longrightarrow FF^*(S)$ by $\widehat{\beta}(\mu) = \beta(\mu)\ \forall \mu \in FKI(S)$. Then by Lemma 54, $\beta(\mu) \in FF^*(S)$. If we denote by $\widehat{\alpha} : FF^*(S) \longrightarrow FKI(S)$ the restriction of α to $FF^*(S)$, by Theorem 55, the compositions $\widehat{\beta} \circ \widehat{\alpha}$ and $\widehat{\alpha} \circ \widehat{\beta}$ are identity mappings. Thus $\widehat{\beta}$ and $\widehat{\alpha}$ are mutually inverse isomorphisms. □

5. Conclusion

In this work, we introduce the concept of kernel fuzzy ideals and $*$-fuzzy filters of a pseudocomplemented semilattice and investigate some of their properties. We give a necessary and sufficient condition for a fuzzy congruence to be a $*$-fuzzy congruence. Furthermore, we identified those fuzzy ideals which can be kernel fuzzy ideals. We also proved that the class of kernel fuzzy ideals form a complete distributive lattice. Moreover, we prove that every fuzzy filter of a pseudocomplemented semilattice is the cokernel of a $*$-fuzzy congruence. Finally, we have shown that there is an isomorphism between the class of $*$-fuzzy filters and the class of kernel fuzzy ideals. Our future work will focus on fuzzy semiprime ideals in general lattices.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The study is partially supported by Bahir Dar University, College of Science.

References

[1] O. Frink, "Pseudo-complements in semi-lattices," *Duke Mathematical Journal*, vol. 29, no. 4, pp. 505–514, 1962.

[2] G. Birkhoff, *Lattice Theory*, vol. 25 of *American Mathematical Society*, Colloquium Publication, New York, NY, USA, 1948.

[3] R. Balbes and A. Horn, "Stone lattices," *Duke Mathematical Journal*, vol. 37, no. 3, pp. 537–545, 1970.

[4] G. Gratzer, "A generalization of stone's representation theorem for boolean algebras," *Duke Mathematical Journal*, vol. 30, pp. 469–474, 1963.

[5] W. H. Cornish, "Congruences on distributive pseudo-complemented lattices," *Bulletin of the Australian Mathematical Society*, vol. 8, no. 2, pp. 161–179, 1973.

[6] T. S. Blyth, "Ideals and filters of pseudo-complemented semilattices," *Proceedings of the Edinburgh Mathematical Society*, vol. 23, no. 3, pp. 301–316, 1980.

[7] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[8] A. Rosenfeld, "Fuzzy subgroups," *Journal of Mathematical Analysis and Applications*, vol. 35, pp. 512–517, 1971.

[9] B. A. Alaba and W. Z. Norahun, "Fuzzy annihilator ideals in distributive lattices," *Annals of Fuzzy Mathematics and Informatics*, vol. 16, no. 2, pp. 191–200, 2018.

[10] B. A. Alaba and W. Z. Norahun, "α-fuzzy ideals and space of prime α-fuzzy ideals in distributive lattices," *Annals of Fuzzy Mathematics and Informatics*, vol. 17, no. 2, pp. 147–163, 2019.

[11] B. A. Alaba and G. M. Addis, "L-fuzzy prime ideals in universal algebras," *Advances in Fuzzy Systems*, vol. 2019, Article ID 5925036, 7 pages, 2019.

[12] B. A. Alaba and G. M. Addis, "L-Fuzzy ideals in universal algebras," *Annals of Fuzzy Mathematics and Informatics*, vol. 17, no. 1, pp. 31–39, 2019.

[13] B. A. Alaba and T. G. Alemayehu, "Closure fuzzy ideals of MS-algebras," *Annals of Fuzzy Mathematics and Informatics*, vol. 16, no. 2, pp. 247–260, 2018.

[14] B. A. Alaba, M. Alamneh, and D. Abeje, "L-fuzzy filters of a poset," *International Journal of Computing Science and Applied Mathematics*, vol. 5, no. 1, pp. 23–29, 2019.

[15] V. Amjid, F. Yousafzai, and K. Hila, "A study of ordered Ag-groupoids in terms of semilattices via smallest (fuzzy) ideals," *Advances in Fuzzy Systems*, vol. 2018, Article ID 8464295, 8 pages, 2018.

[16] O. Kazancı, Ş. Yılmaz, and B. Davvaz, "On fuzzy ordered hyperideals in ordered semihyperrings," *Advances in Fuzzy Systems*, vol. 2019, Article ID 3693926, 7 pages, 2019.

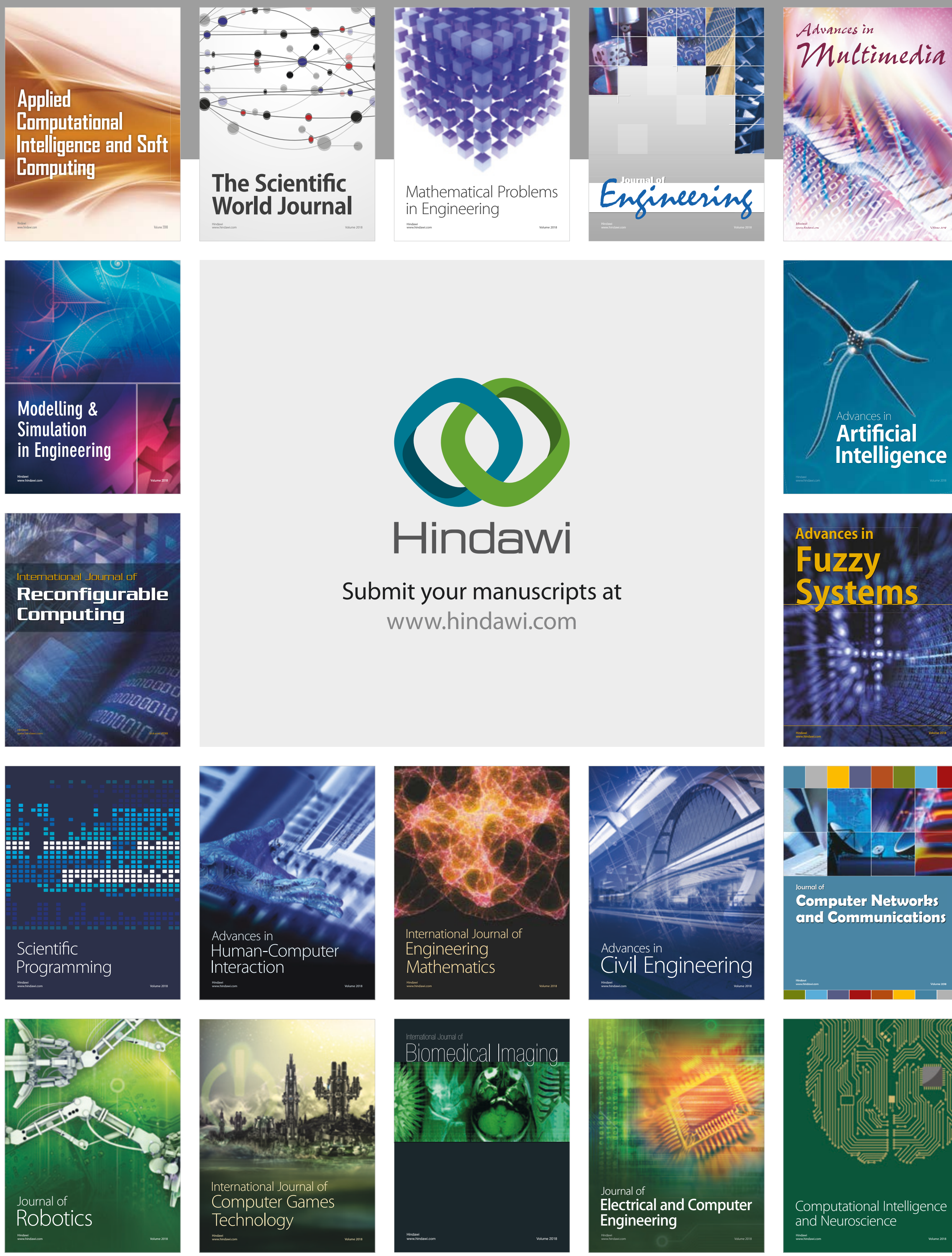
[17] W. J. Liu, "Fuzzy invariant subgroups and fuzzy ideals," *Fuzzy Sets and Systems*, vol. 8, no. 2, pp. 133–139, 1982.

[18] U. M. Swamy and D. V. Raju, "Fuzzy ideals and congruences of lattices," *Fuzzy Sets and Systems*, vol. 95, no. 2, pp. 249–253, 1998.

[19] B. Yuan and W. Wu, "Fuzzy ideals on a distributive lattice," *Fuzzy Sets and Systems*, vol. 35, no. 2, pp. 231–240, 1990.

[20] G. Gratzer, *General Lattice Theory*, Pure and Applied Mathematics, Academic Press, New York, NY, USA, 1978.

[21] T. Ramarao, *Fuzzy ideals and filters of lattice, [Doctoral thesis]*, Department of Mathematics, Acharya Nagarjuna University, Nagar, India, 2011.


Applied Computational Intelligence and Soft Computing
The Scientific World Journal
Mathematical Problems in Engineering
Journal of Engineering
Advances in Multimedia
Modelling & Simulation in Engineering
Hindawi
Submit your manuscripts at
www.hindawi.com
Advances in Artificial Intelligence
International Journal of Reconfigurable Computing
Advances in Fuzzy Systems
Scientific Programming
Advances in Human-Computer Interaction
International Journal of Engineering Mathematics
Advances in Civil Engineering
Journal of Computer Networks and Communications
Journal of Robotics
International Journal of Computer Games Technology
International Journal of Biomedical Imaging
Journal of Electrical and Computer Engineering
Computational Intelligence and Neuroscience