Hindawi
Advances in Fuzzy Systems
Volume 2019, Article ID 4142382, 14 pages
https://doi.org/10.1155/2019/4142382

Research Article

Nonlinear Triangular Intuitionistic Fuzzy Number and Its Application in Linear Integral Equation

Sankar Prasad Mondal,[1] **Adrijit Goswami**,[2] **and Sujit Kumar De**[3]

[1]Department of Natural Science, Maulana Abul Kalam Azad University of Technology, West Bengal, Haringhata, Nadia, West Bengal, India
[2]Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur-721302, India
[3]Department of Mathematics, Midnapore College (Autonomous), Midnapore-721101, West Bengal, India

Correspondence should be addressed to Sankar Prasad Mondal; sankar.res07@gmail.com

Received 22 May 2018; Revised 12 October 2018; Accepted 9 December 2018; Published 3 March 2019

Academic Editor: Ferdinando Di Martino

In this paper we introduce the different arithmetic operations on nonlinear intuitionistic fuzzy number (NIFN). All the arithmetic operations are done by max-min principle method which is nothing but the application of interval analysis. We also define the nonlinear intuitionistic fuzzy function which is used for finding the values, ambiguities, and ranking of nonlinear intuitionistic fuzzy number. The de-i-fuzzification of the corresponding intuitionistic fuzzy solution is done by average of (α, β)-cut method. Finally we solve integral equation with NIFN by the help of intuitionistic fuzzy Laplace transform method.

1. Introduction

1.1. Intuitionistic Fuzzy Sets Theory. Lotfi A. Zadeh [1] published the theory on fuzzy sets and systems. Chang and Zadeh [2] introduced the concept of fuzzy numbers. Different mathematicians have been studying them (dimension of one or dimension of n, see, for example, [3–6]). With the improvements of theories and applications of fuzzy numbers, this concept becomes more and more significant.

Generalization of [1] is taken to be one of intuitionistic fuzzy set (IFS) theory. IFS was first introduced by Atanassov [7] and has been found to be suitable for dealing with various important areas. The fuzzy set considers only the degree of belongingness but not the nonbelongingness. Fuzzy set theory does not incorporate the degree of hesitation (i.e., degree of nondeterminacy defined). To handle such facts, Atanassov [7] explored the concept of fuzzy set theory by IFS theory. The degree of acceptance in fuzzy sets is considered only, but on the other hand IFS is characterized by a membership function and a nonmembership function so that the sum of both values is less than one [8]. Various results on intuitionistic fuzzy set theory are discussed in the papers [9, 10]. The uncertainty theory and calculus constitute a very popular topic nowadays [11–16].

1.2. Fuzzy Integral Equation. Integral equation is very important in the theory of calculus. Nowadays it is very important for application. Now if it is with uncertainty, then its behavior changes. In this paper the idea of intuitionistic fuzzy integral equation is given when the intuitionistic fuzzy number is taken as nonlinear in the membership concept. Before going to the main topic we need to study previous works related to the topic which are done by different researchers. Intuitionistic fuzzy integral is discussed in [17]. There exist several literature sources where fuzzy integral equation is solved such as fuzzy Fredholm integral equation [18–23] and fuzzy Volterra integral equation [24–29].

1.3. Motivation. Many authors consider intuitionistic fuzzy number in different articles and apply it in different areas. But the point is that they considered the intuitionistic fuzzy number with only the linear membership and nonmembership function. But it is not always necessary to consider the membership and nonmembership functions as linear functions. Linear membership and nonmembership function can be a special case. In this paper we consider the intuitionistic fuzzy number with nonlinear membership and nonmembership functions. Previously, many researchers found arithmetic operation on intuitionistic fuzzy number by different methods. Most of them consider the resultant number as an approximated intuitionistic fuzzy number. Now how can we find some operation between two said numbers using interval arithmetic concept? If we consider the number with integral equation, then what is its solution? How can we find approximated crisp value of the intuitionistic fuzzy numbers? Few questions arise on the researcher's mind. From that motivation we try to find the best possible work on this paper.

1.4. Novelties. In spite of the few above-mentioned developments, other few developments can still be done in this paper, which are

(i) formulation of the concept of nonlinear intuitionistic fuzzy number;

(ii) arithmetic operation of nonlinear intuitionistic fuzzy number by max-min principle;

(iii) applying this number with integral equation problem;

(iv) using intuitionistic fuzzy Laplace transform for solving intuitionistic integral equation;

(v) finding the valuation, ambiguities, and ranking of intuitionistic fuzzy function;

(vi) de-i-fuzzification of said number, being done here by average of (α, β)-cut method.

1.5. Structure of the Paper. The structure of the paper is as follows: In the first section we imitate the previously published work on fuzzy and intuitionistic fuzzy integral equations. The second section presents the basic preliminary concept. We define intuitionistic fuzzy Laplace transform and its properties. In the third section we introduce nonlinear intuitionistic fuzzy number and find the arithmetic operation on that number using max-min principle method. The concept of ranking of the number is also addressed in this section. The de-i-fuzzification of the number is done by mean of (α, β)-cut method in the fourth section. The intuitionistic fuzzy distance and integral are defined in the fifth section. The sixth section provides the construction and solution of integral equation in intuitionistic fuzzy environment. The conclusion is given in the seventh section.

2. Preliminaries

2.1. Basic Concept Intuitionistic Fuzzy Set Theory

Definition 1 (intuitionistic fuzzy set: [8]). An IFS $\widetilde{A}^i$ in X is an object having the form $\widetilde{A}^i = \{< x, \mu_{\widetilde{A}^i}(x), \vartheta_{\widetilde{A}^i}(x) >: x \in X\}$, where the $\mu_{\widetilde{A}^i}(x) : X \longrightarrow [0,1]$ and $\vartheta_{\widetilde{A}^i}(x) : X \longrightarrow [0,1]$ define the degree of membership and degree of nonmembership, respectively, of the element $x \in X$ to the set $\widetilde{A}^i$, which is a subset of X, for every element of $x \in X$, $0 \leq \mu_{\widetilde{A}^i}(x) + \vartheta_{\widetilde{A}^i}(x) \leq 1$.

Definition 2 (triangular intuitionistic fuzzy number: [30]). A TIFN $\widetilde{A}^i$ is a subset of IFN in R with following membership function and nonmembership function as follows:

$$\mu_{\widetilde{A}^i}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & for\, a_1 \leq x < a_2, \\ \dfrac{a_3 - x}{a_3 - a_2} & for\, a_2 < x \leq a_3 \\ 0 & otherwise, \end{cases}$$
$$\vartheta_{\widetilde{A}^i}(x) = \begin{cases} \dfrac{a_2 - x}{a_2 - a_1'} & for\, a_1' \leq x < a_2 \\ \dfrac{x - a_2}{a_3' - a_2} & for\, a_2 < x \leq a_3' \\ 1 & otherwise \end{cases} \tag{1}$$

where $a_1' \leq a_2 \leq a_3'$ and $a_1 \leq a_2 \leq a_3$.

The TIFN is denoted by $\widetilde{A}^i_{TIFN} = (a_1, a_2, a_3; a_1', a_2, a_3')$.

Definition 3. Let us consider intuitionistic fuzzy-valued function $f^i(t)$ defined in the parametric form

$$\left[\left[f^i(t)\right]_{(\alpha,\beta)} = \left[f_1(t,\alpha), f_2(t,\alpha); g_1(t,\beta), g_2(t,\beta)\right]\right]. \tag{2}$$

Therefore if we consider the above said intuitionistic fuzzy number, the parametric form is as follows.

$$\left[\widetilde{A}^i\right]_{(\alpha,\beta)} = \left[a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2); a_2 - \beta\left(a_2 - a_1'\right), a_2 + \beta\left(a_3' - a_2\right)\right] \tag{3}$$

2.2. Intuitionistic Fuzzy Laplace Transform. Suppose that $f^i(t)$ is an intuitionistic fuzzy-valued function and s is a real parameter. We define the intuitionistic fuzzy Laplace transform of f as follows.

Definition 4 (see [31]). The intuitionistic fuzzy Laplace transform of an intuitionistic fuzzy-valued function $f^i(t)$ is defined as follows.

$$\boldsymbol{L}\left\{f^i(t)\right\} = \int_0^\infty e^{-st} \odot f^i(t)\,dt = \lim_{\epsilon\longrightarrow\infty} \int_0^\epsilon e^{-st} \odot f^i(t)\,dt, \quad \text{where } s > 0 \tag{4}$$

Consider the intuitionistic fuzzy-valued function $f^i(t)$; then the lower and upper intuitionistic fuzzy Laplace transform of this function are denoted based on the lower and upper intuitionistic fuzzy-valued function $f^i(t)$ as follows.

$\boldsymbol{L}\{f^i(t)\}_{(\alpha,\beta)} = [\lim_{\epsilon\longrightarrow\infty} \int_0^\epsilon e^{-st} f_1(t,\alpha)dt, \lim_{\epsilon\longrightarrow\infty} \int_0^\epsilon e^{-st} f_2(t,\alpha)dt; \lim_{\epsilon\longrightarrow\infty} \int_0^\epsilon e^{-st} g_1(t,\beta)dt, \lim_{\epsilon\longrightarrow\infty} \int_0^\epsilon e^{-st} g_2(t,\beta)dt]$

i.e., $\boldsymbol{L}\{f^i(t)\}_{(\alpha,\beta)} = [l\{f_1(t,\alpha)\}, l\{f_2(t,\alpha)\}; l\{g_1(t,\beta)\}, l\{g_2(t,\beta)\}]$

Now we define the absolute value of an intuitionistic fuzzy-valued function as follows.

Basic Property

(1) **Linearity property:** Let $h_1{}^i(t), h_2{}^i(t)$ be two continuous intuitionistic fuzzy-valued functions; then

$$\boldsymbol{L}(c_1 h_1{}^i(t) \oplus c_2 h_2{}^i(t)) = c_1 \boldsymbol{L}(h_1{}^i(t)) \oplus c_2 \boldsymbol{L}(h_2{}^i(t)).$$

Remark 5. Let $f^i(t)$ be a continuous intuitionistic fuzzy-valued function on $[0,\infty)$ and ≥ 0, then
$\boldsymbol{L}[\rho \odot f^i(t)] = \rho \odot \boldsymbol{L}[f^i(t)]$.

(2) **First Translation Theorem:** Let $f^i(t)$ be a continuous intuitionistic fuzzy-valued function and $\boldsymbol{L}[f^i(t)] = F(s)$, then
$\boldsymbol{L}[f^i(t) \odot e^{at}] = F(s-a)$, where e^{at} is real-valued function and $s-a > 0$.

Definition 6. In order to solve intuitionistic fuzzy differential equations, it is necessary to know the intuitionistic fuzzy Laplace transform of the derivative of $f^i, f^{i'}$. The virtue of $\boldsymbol{L}(f^{i'}(t))$ is that it can be written in terms of $\boldsymbol{L}[f^i(t)]$.

Theorem 7. *Suppose that $f^i(t)$ is continuous intuitionistic fuzzy-valued function on $[0,\infty)$ and exponential order p and that $f^i(t)$ is piecewise continuous intuitionistic fuzzy-valued function on $[0,\infty)$; then*

(a) $\boldsymbol{L}(f^{i'}(t)) = s\boldsymbol{L}[f^i(t)] \ominus f^i(0)$, *if $f^i(t)$ is (i)-differentiable,*

(b) $\boldsymbol{L}(f^{i'}(t)) = (-f^i(0)) \ominus (-s\boldsymbol{L}(f^i(t))$, *if $f^i(t)$ is (ii)-differentiable.*

Proof. (a) **L.H.S:** $\boldsymbol{L}(f^{i'}(t)) = [f_1'(t,\alpha), f_2'(t,\alpha); g_1'(t,\beta), g_2'(t,\beta)]$
and **R.H.S:** $s\boldsymbol{L}[f^i(t)] \ominus f^i(0)$
$= s[l\{f_1(t,\alpha)\}, l\{f_2(t,\alpha)\}; l\{g_1(t,\beta)\}, l\{g_2(t,\beta)\}] \ominus [f_1(0,\alpha), f_2(0,\alpha); g_1(0,\beta), \{g_2(0,\beta)\}]$
$= [sl\{f_1(t,\alpha)\} - f_1(0,\alpha), sl\{f_2(t,\alpha)\} - f_2(0,\alpha); sl\{g_1(t,\beta)\} - g_1(0,\beta), sl\{g_2(t,\beta)\} - g_2(0,\beta)]$
$= [f_1'(t,\alpha), f_2'(t,\alpha); g_1'(t,\beta), g_2'(t,\beta)]$.
Hence, **L.H.S=R.H.S**. □

Proof. (b) **L.H.S:** $\boldsymbol{L}(f^{i'}(t)) = [f_2'(t,\alpha), f_1'(t,\alpha); g_2'(t,\beta), g_1'(t,\beta)]$ and
R.H.S: $(-f^i(0)) \ominus (-s\boldsymbol{L}(f^i(t))$
$=(-)[f_1(0,\alpha), f_2(0,\alpha); g_1(0,\beta), g_2(0,\beta)] \ominus (-s)l[f_1(t,\alpha), f_2(t,\alpha); g_1(t,\beta), g_2(t,\beta)]$
$= [-f_2(0,\alpha) + sl\{f_2(t,\alpha)\}, -f_1(0,\alpha) + sl\{f_1(t,\alpha)\}; -g_2(0,\beta) + sl\{g_2(t,\beta)\}, -g_1(0,\beta) + sl\{g_1(t,\beta)\}] = [f_2'(t,\alpha), f_1'(t,\alpha); g_2'(t,\beta), g_1'(t,\beta)] = \boldsymbol{L}(f^{i'}(t))$.
Hence, **L.H.S=R.H.S**. □

3. Nonlinear Triangular Intuitionistic Fuzzy Number and Its Arithmetic Operations

Definition 8 (see [32]). A NTIFN $\widetilde{A}^{ni}$ is a subset of IFN in R with the following membership function and nonmembership function:

$$\mu_{\widetilde{A}^{ni}}(x) = \begin{cases} \left(\dfrac{x-a_1}{a_2-a_1}\right)^p & \text{for } a_1 \leq x < a_2 \\ 1 & \text{for } x = a_2 \\ \left(\dfrac{a_3-x}{a_3-a_2}\right)^q & \text{for } a_2 < x \leq a_3 \\ 0 & \text{otherwise,} \end{cases}$$
$$\vartheta_{\widetilde{A}^{ni}}(x_1) = \begin{cases} \left(\dfrac{a_2-x}{a_2-a_1'}\right)^r & \text{for } a_1' \leq x < a_2 \\ 0 & \text{for } x = a_2 \\ \left(\dfrac{x-a_2}{a_3'-a_2}\right)^s & \text{for } a_2 < x \leq a_3' \\ 1 & \text{otherwise} \end{cases} \quad (5)$$

where $a_1' \leq a_2 \leq a_3'$ and $a_1 \leq a_2 \leq a_3$.

The TIFN is denoted by $\widetilde{A}^{ni}{}_{TIFN} = (a_1, a_2, a_3; a_1', a_2, a_3')$.

Definition 9 (α-cut set). **A** α-cut set of $\widetilde{A}^{ni}{}_{TIFN} = (a_1, a_2, a_3; a_1', a_2, a_3')$ is a crisp subset of R which is defined as follows.

$$\begin{aligned} A_\alpha &= \left\{x : \mu_{\widetilde{A}^i}(x) \geq \alpha,\ \forall \alpha \in [0,1]\right\} \\ &= \left[A_1(\alpha), A_2(\alpha)\right] \\ &= \left[a_1 + \alpha^{1/p}(a_2-a_1), a_3 - \alpha^{1/q}(a_3-a_2)\right] \end{aligned} \quad (6)$$

Definition 10 (β-cut set). A β-cut set of $\widetilde{A}^{ni}{}_{TIFN} = (a_1, a_2, a_3; a_1', a_2, a_3')$ is a crisp subset of R which is defined as follows.

$$\begin{aligned} A_\beta &= \left\{x : \vartheta_{\widetilde{A}^i}(x) \leq \beta,\ \forall \beta \in [0,1]\right\} \\ &= \left[A_1'(\beta), A_2'(\beta)\right] \\ &= \left[a_2 - \beta^{1/r}(a_2-a_1'), a_2 + \beta^{1/s}(a_3'-a_2)\right] \end{aligned} \quad (7)$$

Definition 11 ((α,β)-cut set). **A** (α,β)-cut set of $\widetilde{A}^i{}_{GTrIFN} = \widetilde{A}^{ni}{}_{TIFN} = (a_1, a_2, a_3; a_1', a_2, a_3')$ is a crisp subset of R which is defined as follows.

$$A_{\alpha,\beta} = \left\{\left[A_1(\alpha), A_2(\alpha)\right]; \left[A_1'(\beta), A_2'(\beta)\right]\right\}, \quad \alpha+\beta \leq 1,\ \alpha \in [0,1],\ \beta \in [0,1] \quad (8)$$

Theorem 12. *The sum of the membership and the nonmembership function at any particular point is between 0 and 1.*

That is, if for a nonlinear intuitionistic fuzzy number $\widetilde{A}^{ni}{}_{TIFN} = (a_1, a_2, a_3; a_1', a_2, a_3')$ (see Figure 1), membership and nonmembership function are denoted by $\mu_{\widetilde{A}^{ni}}(x)$ and $\vartheta_{\widetilde{A}^{ni}}(x_1)$; then
$0 \leq \mu_{\widetilde{A}^{ni}}(x) + \vartheta_{\widetilde{A}^{ni}}(x_1) \leq 1$.

Proof. From Figure 1 we prove the theorem by splitting up the region.

Table 1: Membership and nonmembership value for different region.

Interval	$\mu_{\widetilde{A}^{ni}}(x)$	$\vartheta_{\widetilde{A}^{ni}}(x_1)$	$\mu_{\widetilde{A}^{ni}}(x)+\vartheta_{\widetilde{A}^{ni}}(x_1)$	Conclusion
$x=a_1'$	0	$\leq\beta$	$\leq\beta$	≤ 1
$a_1'<x<a_1$	0	$<\beta$	$<\beta$	<1
$x=a_1$	0	$\leq\beta$	$\leq\beta$	≤ 1
$a_1<x<a_2$	$<\alpha$	$<\beta$	$<\alpha+\beta$	≤ 1
$x=a_2$	α	0	α	1
$a_2<x<a_3$	$<\alpha$	$<\beta$	$<\alpha+\beta$	≤ 1
$x=a_3$	0	$<\beta$	$<\beta$	≤ 1
$a_3<x<a_3'$	0	$<\beta$	$<\alpha+\beta$	≤ 1
$x=a_3'$	0	$\leq\beta$	$\leq\beta$	≤ 1

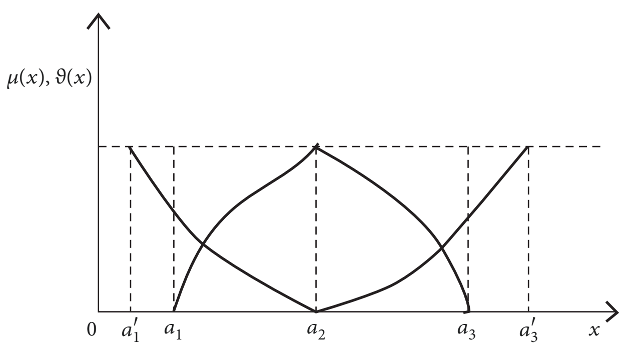

Figure 1: Nonlinear intuitionistic fuzzy number.

Now we split up the region into different intervals and points as

$$a_1', [a_1', a_1], a_1, [a_1, a_2], a_2, [a_2, a_3], a_3, [a_3, a_3'], a_3' \quad (9)$$

and first take $n = 1$. Then in each of the above points and intervals, the values of membership function and nonmembership function are assumed as in Table 1. □

Note 13. The above proof is done for taking $n = 1$. It can be proved by all values of n. For other values other than 1, we can take numerical examples for proving the theorem.

3.1. Max-Min Principle Method for Arithmetic Operation on Intuitionistic Fuzzy Number. Let $\widetilde{A}^{ni}$ and $\widetilde{B}^{ni}$ be two fuzzy numbers and $*$ be an arbitrary operation such that

$$\widetilde{C}^{ni} = \widetilde{A}^{ni} * \widetilde{B}^{ni}. \quad (10)$$

We know that, for any arbitrary operation between two fuzzy numbers, the resulting fuzzy numbers are not the same as the typed fuzzy numbers in nature. Many authors consider the approximated resulting fuzzy number.

Our aim is to first convert the fuzzy number into parametric fuzzy number, and using interval arithmetic operation we find the resulting fuzzy number in parametric form.

Let (α,β)-cut of $\widetilde{C}^{ni}$ be $[c_1(\alpha), c_2(\alpha); c_1'(\beta), c_1'(\beta)]$, $\widetilde{A}^{ni}$ be $[a_1(\alpha), a_2(\alpha); a_1'(\beta), a_2'(\beta)]$, and $\widetilde{B}^{ni}$ be $[b_1(\alpha), b_2(\alpha); b_1'(\beta), b_2'(\beta)]$.

Now the component of the resulting fuzzy number in parametric form is written as

$$\begin{aligned}
c_1(\alpha) &= \min\{a_1(\alpha) * b_1(\alpha), a_1(\alpha) * b_2(\alpha), a_2(\alpha) \\
&\quad * b_1(\alpha), a_2(\alpha) * b_2(\alpha)\} \\
c_2(\alpha) &= \max\{a_1(\alpha) * b_1(\alpha), a_1(\alpha) * b_2(\alpha), a_2(\alpha) \\
&\quad * b_1(\alpha), a_2(\alpha) * b_2(\alpha)\} \\
c_1'(\beta) &= \min\{a_1'(\beta) * b_1'(\beta), a_1'(\beta) * b_2'(\beta), a_2'(\beta) \\
&\quad * b_1'(\beta), a_2'(\beta) * b_2'(\beta)\} \\
c_1'(\beta) &= \max\{a_1'(\beta) * b_1'(\beta), a_1'(\beta) * b_2'(\beta), a_2'(\beta) \\
&\quad * b_1'(\beta), a_2'(\beta) * b_2'(\beta)\}
\end{aligned} \quad (11)$$

where $c_1(\alpha), c_1'(\beta)$ are increasing functions and $c_2(\alpha), c_1'(\beta)$ are decreasing functions.

3.2. Arithmetic Operation on Nonlinear Intuitionistic Fuzzy Number. If $\widetilde{A}^{ni}_{TIFN} = (a_1, a_2, a_3; a_1', a_2, a_3')$ and $\widetilde{B}^{ni}_{TIFN} = (b_1, b_2, b_3; b_1', b_2, b_3')$ are two nonlinear triangular fuzzy numbers with the membership function and the nonmembership function as

$$\mu_{\widetilde{A}^{ni}}(x) = \begin{cases} \left(\dfrac{x-a_1}{a_2-a_1}\right)^{p_1} & \textit{for } a_1 \leq x < a_2 \\ 1 & \textit{for } x = a_2 \\ \left(\dfrac{a_3-x}{a_3-a_2}\right)^{q_1} & \textit{for } a_2 < x \leq a_3 \\ 0 & \textit{otherwise}, \end{cases}$$
$$\vartheta_{\widetilde{A}^{ni}}(x) = \begin{cases} \left(\dfrac{a_2-x}{a_2-a_1'}\right)^{r_1} & \textit{for } a_1' \leq x < a_2 \\ 0 & \textit{for } x = a_2 \\ \left(\dfrac{x-a_2}{a_3'-a_2}\right)^{s_1} & \textit{for } a_2 < x \leq a_3' \\ 1 & \textit{otherwise} \end{cases} \quad (12)$$

and

$$\mu_{\widetilde{B}^{ni}}(x)=\begin{cases}\left(\frac{x-b_1}{b_2-b_1}\right)^{p_2} & for\, b_1\le x<b_2\\ 1 & for\, x=b_2\\ \left(\frac{b_3-x}{b_3-b_2}\right)^{q_2} & for\, b_2<x\le b_3\\ 0 & otherwise,\end{cases} \tag{13}$$

$$\vartheta_{\widetilde{B}^{ni}}(x)=\begin{cases}\left(\frac{b_2-x}{b_2-b_1'}\right)^{r_2} & for\ b_1'\le x<b_2\\ 0 & for\ x=b_2\\ \left(\frac{x-b_2}{b_3'-b_2}\right)^{s_2} & for\ b_2<x\le b_3'\\ 1 & otherwise\end{cases} \tag{14}$$

with (α,β)-cut

$$\begin{aligned}A_{\alpha,\beta}=\{&[a_1+\alpha^{1/p_1}(a_2-a_1),a_3-\alpha^{1/q_1}(a_3-a_2)];\\&[a_2-\beta^{1/r_1}(a_2-a_1'),a_2+\beta^{1/s_1}(a_3'-a_3)]\}\end{aligned} \tag{15}$$

and

$$\begin{aligned}B_{\alpha,\beta}=\{&[b_1+\alpha^{1/p_2}(b_2-b_1),b_3-\alpha^{1/q_2}(b_3-b_2)];\\&[b_2-\beta^{1/r_2}(b_2-b_1'),b_2+\beta^{1/s_2}(b_3'-b_3)]\}\end{aligned} \tag{16}$$

for a particular case we consider that $p_1=p_2=q_1=q_2=n_1$ and $r_1=r_2=s_1=s_2=n_2$.

3.2.1. Addition of Two Normal Fuzzy Numbers Using (α,β)-Cut

$$\begin{aligned}C_{\alpha,\beta}=A_{\alpha,\beta}+B_{\alpha,\beta}=\{&[a_1+b_1\\&+\alpha^{1/n_1}(a_2+b_2-a_1-b_1),a_3+b_3\\&-\alpha^{1/n_1}(a_3+b_3-a_2-b_2)];[a_2+b_2\\&-\beta^{1/n_2}(a_2+b_2-a_1'-b_1'),a_2+b_2\\&+\beta^{1/n_2}(a_3'+b_3'-a_2-b_2)]\}\end{aligned} \tag{17}$$

The membership and the nonmembership function are defined as of $\widetilde{C}^{ni}{}_{TIFN}$

$$\mu_{\widetilde{B}^{ni}}(x)=\begin{cases}\left(\frac{x-a_1-b_1}{a_2+b_2-a_1-b_1}\right)^{n_1} & for\ a_1+b_1\le x<a_2+b_2\\ 1 & for\ x=a_2+b_2\\ \left(\frac{a_3+b_3-x}{a_3+b_3-a_2-b_2}\right)^{n_1} & for\ a_2+b_2<x\le a_3+b_3\\ 0 & otherwise,\end{cases} \tag{18}$$

and

$$\vartheta_{\widetilde{B}^{ni}}(x)=\begin{cases}\left(\frac{a_2+b_2-x}{a_2+b_2-a_1'-b_1'}\right)^{n_2} & for\ a_1'+b_1'\le x<a_2+b_2\\ 0 & for\ x=a_2+b_{22}\\ \left(\frac{x-a_2-b_2}{a_3'+b_3'-a_2-b_2}\right)^{n_2} & for\ a_2+b_2<x\le a_3'+b_3'\\ 1 & otherwise.\end{cases} \tag{19}$$

3.2.2. Subtraction of Two Normal Fuzzy Numbers Using (α,β)-Cut

$$\begin{aligned}C_{\alpha,\beta}=A_{\alpha,\beta}-B_{\alpha,\beta}=\{&[a_1-b_3\\&+\alpha^{1/n_1}(a_2-b_2-a_1+b_3),a_3-b_1\\&-\alpha^{1/n_1}(a_3-b_1-a_2+b_2)];[a_2-b_2\\&-\beta^{1/n_2}(a_2-b_2-a_1'+b_3'),a_2-b_2\\&+\beta^{1/n}(a_3'-b_1'-a_2+b_2)]\}\end{aligned} \tag{20}$$

The membership and the nonmembership function are defined as for $\widetilde{C}^{ni}{}_{TIFN}$

$$\mu_{\widetilde{B}^{ni}}(x)=\begin{cases}\left(\frac{x-a_1+b_3}{a_2-b_2-a_1+b_3}\right)^{n_1} & for\ a_1-b_3\le x<a_2-b_2\\ 1 & for\ x=a_2-b_2\\ \left(\frac{a_3-b_1-x}{a_3-b_1-a_2-b_2}\right)^{n_1} & for\ a_2-b_2<x\le a_3-b_1\\ 0 & otherwise,\end{cases} \tag{21}$$

and

$$\vartheta_{\widetilde{B}^{ni}}(x)=\begin{cases}\left(\frac{a_2-b_2-x}{a_2-b_2-a_1'+b_3'}\right)^{n_2} & for\, a_1'-b_3'\le x<a_2-b_2\\ 0 & for\, x=a_2-b_2\\ \left(\frac{x-a_2+b_2}{a_3'-b_1'-a_2+b_2}\right)^{n_2} & for\, a_2-b_2<x\le a_3'-b_1'\\ 1 & otherwise.\end{cases} \tag{22}$$

3.2.3. Multiplication by a Scalar. If $k>0$ is a positive scalar, then

$$\begin{aligned}kA_{\alpha,\beta}=\{&[k(a_1+\alpha^{1/n_1}(a_2-a_1)),\\&k(a_3-\alpha^{1/n_1}(a_3-a_2))];[k(a_2-\beta^{1/n_2}(a_2-a_1')),\\&k(a_2+\beta^{1/n_2}(a_3'-a_3))]\}.\end{aligned} \tag{23}$$

Therefore, $k\widetilde{A}^{ni}{}_{TIFN}=(ka_1,ka_2,ka_3;ka_1',ka_2,ka_3')$.

If $k < 0$ is a negative scalar, then

$$kA_{\alpha,\beta} = \left\{\left[k\left(a_3 - \alpha^{1/n_1}\left(a_3 - a_2\right)\right),\right.\right.
k\left(a_1 + \alpha^{1/n_1}\left(a_2 - a_1\right)\right)\right];\left[k\left(a_2 + \beta^{1/n_2}\left(a_3' - a_3\right)\right),
\left.\left.k\left(a_2 - \beta^{1/n_2}\left(a_2 - a_1'\right)\right)\right]\right\}. \quad (24)$$

Therefore, $k\widetilde{A}^{ni}_{TIFN} = (ka_3, ka_2, ka_1; ka_3', ka_2, ka_1')$.

3.2.4. Multiplication and Division of Two Nonlinear Fuzzy Numbers Using Interval Rule Base Method. Consider two intervals $[a, b]$ and $[c, d]$

where a, b, c, d may be positive or negative.

Therefore we can write $[a,b] \bullet [c,d] = [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}]$ and $[a,b]/[c,d] = [\min\{a/c, a/d, b/c, b/d\}, \max\{a/c, a/d, b/c, b/d\}]$.

For intuitionistic fuzzy multiplication or division we can use this concept on the (α, β)-cut interval.

$$\left[A_1(\alpha), A_2(\alpha)\right]; \left[A_1'(\beta), A_2'(\beta)\right] \quad (25)$$

Multiplication: $C_{\alpha,\beta} = A_{\alpha,\beta} \bullet B_{\alpha,\beta} = [P_1(\alpha), P_2(\alpha); P_1'(\beta), P_2'(\beta)]$

where

$$\begin{aligned}
P_1(\alpha) &= \min\{A_1(\alpha)B_1(\alpha), A_1(\alpha)B_2(\alpha), A_2(\alpha) \cdot B_1(\alpha), A_2(\alpha)B_2(\alpha)\} \\
P_2(\alpha) &= \max\{A_1(\alpha)B_1(\alpha), A_1(\alpha)B_2(\alpha), A_2(\alpha) \cdot B_1(\alpha), A_2(\alpha)B_2(\alpha)\} \\
P_1'(\beta) &= \min\{A_1'(\beta)B_1'(\beta), A_1'(\beta)B_2'(\beta), A_2'(\beta) \cdot B_1'(\beta), A_2'(\beta)B_2'(\beta)\} \\
P_2'(\beta) &= \max\{A_1'(\beta)B_1'(\beta), A_1'(\beta)B_2'(\beta), A_2'(\beta) \cdot B_1'(\beta), A_2'(\beta)B_2'(\beta)\}
\end{aligned} \quad (26)$$

Division: $C_{\alpha,\beta} = A_{\alpha,\beta}/B_{\alpha,\beta} = [Q_1(\alpha), Q_2(\alpha); Q_1'(\beta), Q_2'(\beta)]$

where

$$\begin{aligned}
P_1(\alpha) &= \min\left\{\frac{A_1(\alpha)}{B_1(\alpha)}, \frac{A_1(\alpha)}{B_2(\alpha)}, \frac{A_2(\alpha)}{B_1(\alpha)}, \frac{A_2(\alpha)}{B_2(\alpha)}\right\} \\
P_2(\alpha) &= \max\left\{\frac{A_1(\alpha)}{B_1(\alpha)}, \frac{A_1(\alpha)}{B_2(\alpha)}, \frac{A_2(\alpha)}{B_1(\alpha)}, \frac{A_2(\alpha)}{B_2(\alpha)}\right\} \\
P_1'(\beta) &= \min\left\{\frac{A_1'(\beta)}{B_1'(\beta)}, \frac{A_1'(\beta)}{B_2'(\beta)}, \frac{A_2'(\beta)}{B_1'(\beta)}, \frac{A_2'(\beta)}{B_2'(\beta)}\right\} \\
P_2'(\beta) &= \max\left\{\frac{A_1'(\beta)}{B_1'(\beta)}, \frac{A_1'(\beta)}{B_2'(\beta)}, \frac{A_2'(\beta)}{B_1'(\beta)}, \frac{A_2'(\beta)}{B_2'(\beta)}\right\}
\end{aligned} \quad (27)$$

For every $\alpha, \beta \in [0, 1]$

$$\begin{aligned}
C_{\alpha,\beta} &= A_{\alpha,\beta} \bullet B_{\alpha,\beta} \\
&== [\min R(\alpha), \max R(\alpha); \min S(\alpha), \max S(\alpha)]
\end{aligned} \quad (28)$$

where $R(\alpha) = [A_1(\alpha)B_1(\alpha), A_1(\alpha)B_2(\alpha), A_2(\alpha)B_1(\alpha), A_2(\alpha)B_2(\alpha)]$ and $S(\alpha) = [A_1'(\beta)B_1'(\beta), A_1'(\beta)B_2'(\beta), A_2'(\beta)B_1'(\beta), A_2'(\beta)B_2'(\beta)]$.

Example 14. If $\widetilde{a} = (40, 45, 50; 38, 45, 52)$ and $\widetilde{b} = (-14, -10, -7; -12, -10, -8)$, then find $\widetilde{a} \bullet \widetilde{b}$ and $\widetilde{a}/\widetilde{b}$.

Solution

$$\begin{aligned}
[\widetilde{a}]_{\alpha,\beta} &= \left[40 + 5\alpha^{1/3}, 50 - 5\alpha^{1/3}; 45 - 7\beta^{1/3}, 45 + 7\beta^{1/3}\right] \\
\left[\widetilde{b}\right]_{\alpha,\beta} &= \left[-14 + 4\alpha^{1/3}, -7 - 3\alpha^{1/3}; -10 - 2\beta^{1/3}, -10 + 2\beta^{1/3}\right]
\end{aligned} \quad (29)$$

Now let $\widetilde{e} = \widetilde{a} \bullet \widetilde{b}$, $\widetilde{h} = \widetilde{a}/\widetilde{b}$, $[\widetilde{e}]_{\alpha,\beta} = [e_1(\alpha), e_2(\alpha); e_1'(\beta), e_2'(\beta)]$, and $[\widetilde{h}]_{\alpha,\beta} = [h_1(\alpha), h_2(\alpha); h_1'(\beta), h_2'(\beta)]$.

Now by interval rule base system we find the following.

$$\begin{aligned}
e_1(\alpha) &= \min\left\{\left(40 + 5\alpha^{1/3}\right) \cdot \left(-14 + 4\alpha^{1/3}\right), \left(40 + 5\alpha^{1/3}\right) \cdot \left(-7 - 3\alpha^{1/3}\right), \left(50 - 5\alpha^{1/3}\right) \cdot \left(-14 + 4\alpha^{1/3}\right), \left(50 - 5\alpha^{1/3}\right)\left(-7 - 3\alpha^{1/3}\right)\right\} \\
e_2(\alpha) &= \max\left\{\left(40 + 5\alpha^{1/3}\right) \cdot \left(-14 + 4\alpha^{1/3}\right), \left(40 + 5\alpha^{1/3}\right) \cdot \left(-7 - 3\alpha^{1/3}\right), \left(50 - 5\alpha^{1/3}\right) \cdot \left(-14 + 4\alpha^{1/3}\right), \left(50 - 5\alpha^{1/3}\right)\left(-7 - 3\alpha^{1/3}\right)\right\} \\
e_1'(\beta) &= \min\left\{\left(45 - 7\beta^{1/3}\right) \cdot \left(-10 - 2\beta^{1/3}\right), \left(45 - 7\beta^{1/3}\right) \cdot \left(-10 + 2\beta^{1/3}\right), \left(45 + 7\beta^{1/3}\right) \cdot \left(-10 - 2\beta^{1/3}\right), \left(45 + 7\beta^{1/3}\right)\left(-10 + 2\beta^{1/3}\right)\right\} \\
e_2'(\beta) &= \max\left\{\left(45 - 7\beta^{1/3}\right) \cdot \left(-10 - 2\beta^{1/3}\right), \left(45 - 7\beta^{1/3}\right) \cdot \left(-10 + 2\beta^{1/3}\right), \left(45 + 7\beta^{1/3}\right) \cdot \left(-10 - 2\beta^{1/3}\right), \left(45 + 7\beta^{1/3}\right)\left(-10 + 2\beta^{1/3}\right)\right\}
\end{aligned} \quad (30)$$

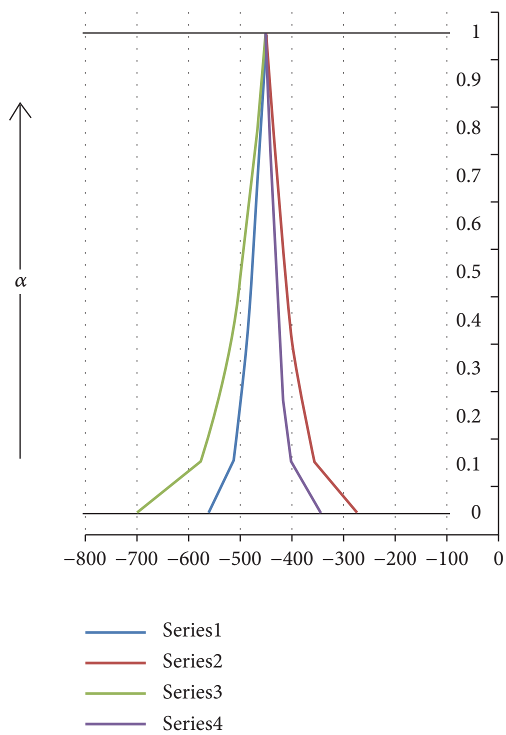

FIGURE 2: Possible $e_1(\alpha)$ and $e_2(\alpha)$ as $\alpha \in [0,1]$.

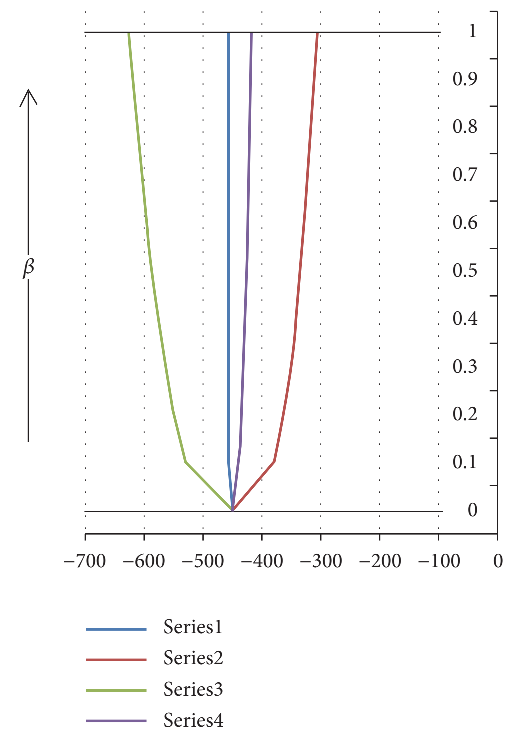

FIGURE 3: Possible $e_1'(\beta)$ and $e_2'(\beta)$ as $\beta \in [0,1]$.

From Figure 2 we can conclude that the $e_1(\alpha)$ and $e_2(\alpha)$ should be chosen as follows.

$$\begin{aligned} e_1(\alpha) &= \left(50 - 5\alpha^{1/3}\right)\left(-14 + 4\alpha^{1/3}\right), \\ e_2(\alpha) &= \left(40 + 5\alpha^{1/3}\right)\left(-7 - 3\alpha^{1/3}\right) \end{aligned} \tag{31}$$

From Figure 3 we can conclude that the $e_1'(\beta)$ and $e_2'(\beta)$ should be chosen as the following.

$$\begin{aligned} e_1'(\beta) &= \left(45 + 7\beta^{1/3}\right)\left(-10 - 2\beta^{1/3}\right), \\ e_2'(\beta) &= \left(45 - 7\beta^{1/3}\right)\left(-10 + 2\beta^{1/3}\right) \end{aligned} \tag{32}$$

Hence by interval rule base the (α,β)-cut of $\tilde{e}$ is given by the following.

$$\begin{aligned} [\tilde{e}]_{\alpha,\beta} = \Big[&\left(50 - 5\alpha^{1/3}\right)\left(-14 + 4\alpha^{1/3}\right), \left(40 + 5\alpha^{1/3}\right) \\ &\cdot \left(-7 - 3\alpha^{1/3}\right); \left(45 + 7\beta^{1/3}\right) \\ &\cdot \left(-10 - 2\beta^{1/3}\right), \left(45 - 7\beta^{1/3}\right)\left(-10 + 2\beta^{1/3}\right)\Big] \end{aligned} \tag{33}$$

Similarly,

$$\begin{aligned} h_1(\alpha) &= \min\left\{\frac{\left(40 + 5\alpha^{1/3}\right)}{\left(-14 + 4\alpha^{1/3}\right)}, \frac{\left(40 + 5\alpha^{1/3}\right)}{\left(-7 - 3\alpha^{1/3}\right)}, \frac{\left(50 - 5\alpha^{1/3}\right)}{\left(-14 + 4\alpha^{1/3}\right)}, \frac{\left(50 - 5\alpha^{1/3}\right)}{\left(-7 - 3\alpha^{1/3}\right)}\right\} \\ h_2(\alpha) &= \max\left\{\frac{\left(40 + 5\alpha^{1/3}\right)}{\left(-14 + 4\alpha^{1/3}\right)}, \frac{\left(40 + 5\alpha^{1/3}\right)}{\left(-7 - 3\alpha^{1/3}\right)}, \frac{\left(50 - 5\alpha^{1/3}\right)}{\left(-14 + 4\alpha^{1/3}\right)}, \frac{\left(50 - 5\alpha^{1/3}\right)}{\left(-7 - 3\alpha^{1/3}\right)}\right\} \\ h_1'(\beta) &= \min\left\{\frac{\left(45 - 7\beta^{1/3}\right)}{\left(-10 - 2\beta^{1/3}\right)}, \frac{\left(45 - 7\beta^{1/3}\right)}{\left(-10 + 2\beta^{1/3}\right)}, \frac{\left(45 + 7\beta^{1/3}\right)}{\left(-10 - 2\beta^{1/3}\right)}, \frac{\left(45 + 7\beta^{1/3}\right)}{\left(-10 + 2\beta^{1/3}\right)}\right\} \\ h_2'(\beta) &= \max\left\{\frac{\left(45 - 7\beta^{1/3}\right)}{\left(-10 - 2\beta^{1/3}\right)}, \frac{\left(45 - 7\beta^{1/3}\right)}{\left(-10 + 2\beta^{1/3}\right)}, \frac{\left(45 + 7\beta^{1/3}\right)}{\left(-10 - 2\beta^{1/3}\right)}, \frac{\left(45 + 7\beta^{1/3}\right)}{\left(-10 + 2\beta^{1/3}\right)}\right\}. \end{aligned} \tag{34}$$

From Figure 5 we can conclude that the $h_1(\alpha)$ and $h_2(\alpha)$ should be chosen as follows.

$$\begin{aligned} h_1(\alpha) &= \frac{\left(50 - 5\alpha^{1/3}\right)}{\left(-7 - 3\alpha^{1/3}\right)}, \\ h_2(\alpha) &= \frac{\left(40 + 5\alpha^{1/3}\right)}{\left(-14 + 4\alpha^{1/3}\right)} \end{aligned} \tag{35}$$

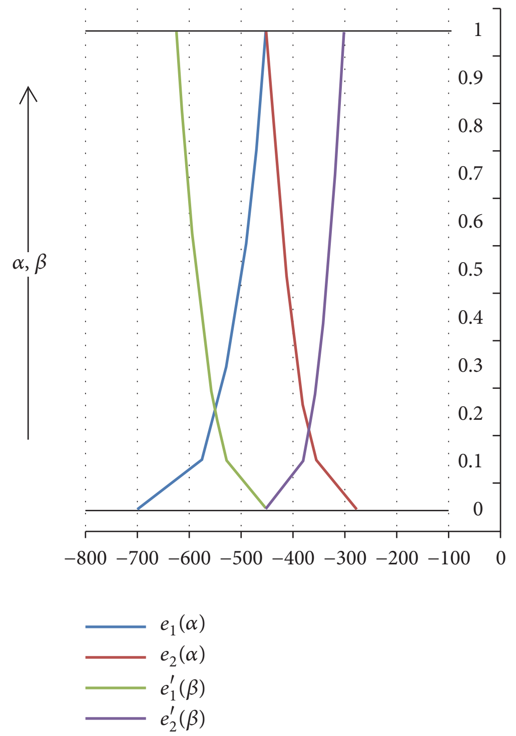

FIGURE 4: $e_1(\alpha)$, $e_2(\alpha)$, $e_1'(\beta)$, and $e_2'(\beta)$ for $\alpha, \beta \in [0, 1]$.

From Figure 6 we can conclude that the $h_1'(\beta)$ and $h_2'(\beta)$ should be chosen as

$$h_1'(\beta) = \frac{(45 + 7\beta^{1/3})}{(-10 + 2\beta^{1/3})} \tag{36}$$

and

$$h_2'(\beta) = \frac{(45 - 7\beta^{1/3})}{(-10 - 2\beta^{1/3})}. \tag{37}$$

Hence by interval rule base the (α, β)-cut of $\tilde{e}$ is given by the following.

$$[\tilde{h}]_{\alpha,\beta} = \left[\frac{(50 - 5\alpha^{1/3})}{(-7 - 3\alpha^{1/3})}, \frac{(40 + 5\alpha^{1/3})}{(-14 + 4\alpha^{1/3})}; \frac{(45 + 7\beta^{1/3})}{(-10 + 2\beta^{1/3})}, \frac{(45 - 7\beta^{1/3})}{(-10 - 2\beta^{1/3})} \right] \tag{38}$$

Remark. We recommend seeing graphical representation Figures 2, 3, 4, 5, 6, and 7

3.3. Intuitionistic Fuzzy Function. Considering that $G_1(\alpha)$, $G_2(\alpha)$, $G_1'(\beta)$, and $G_2'(\beta)$ are the continuous functions on the interval I.

The set $\tilde{G}$ can be determined by membership and non-membership functions as follows:

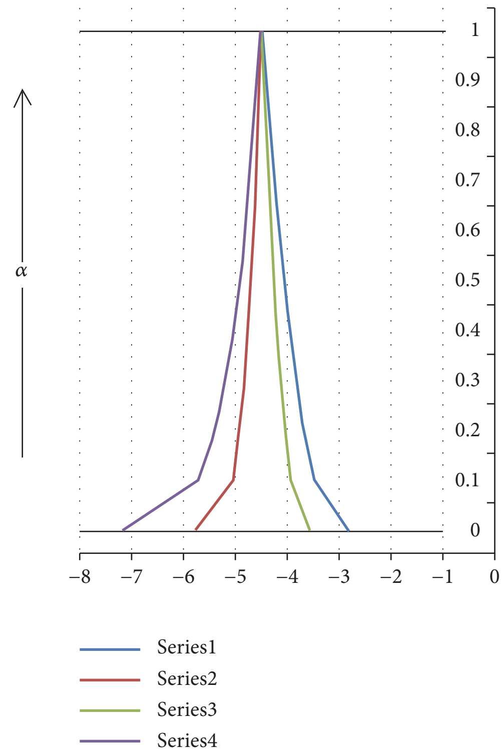

FIGURE 5: Possible $h_1(\alpha)$ and $h_2(\alpha)$ as $\alpha \in [0, 1]$.

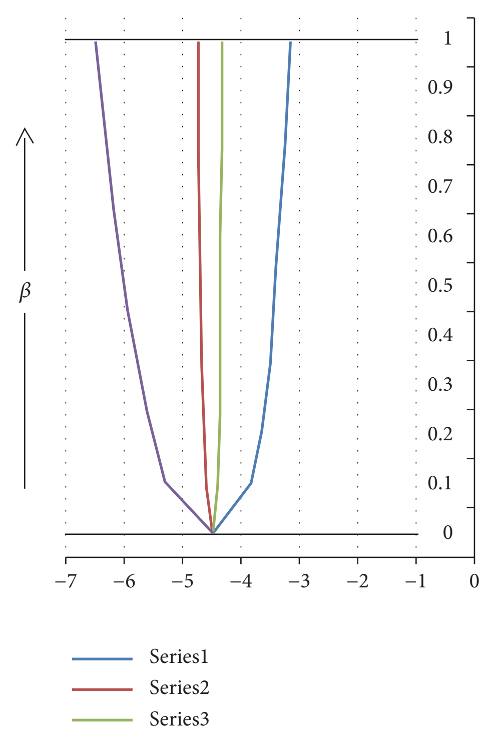

FIGURE 6: Possible $h_1'(\beta)$ and $h_2'(\beta)$ as $\beta \in [0, 1]$.

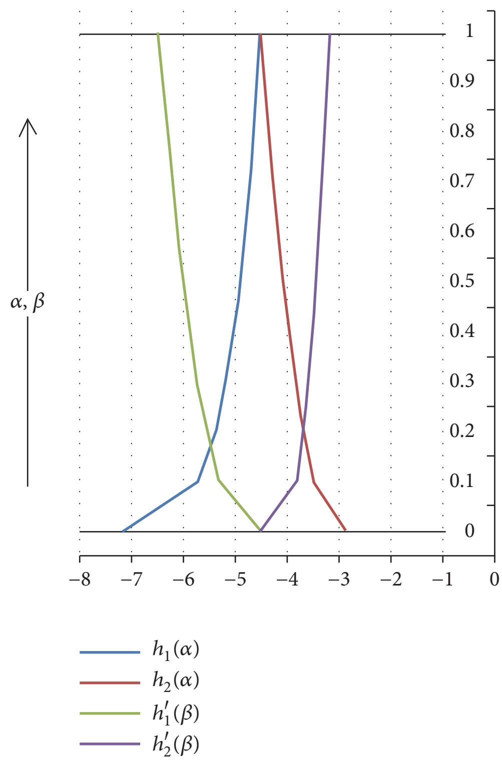

Figure 7: $h_1(\alpha)$, $h_2(\alpha)$, $h_1'(\beta)$, and $h_2'(\beta)$ for $\alpha, \beta \in [0,1]$.

$$\mu_{\widetilde{G}(t)}(m(\alpha)) = \begin{cases} \alpha, m(\alpha) = G_1(\alpha), & 0 \le \alpha \le 1 \\ \alpha, m(\alpha) = G_2(\alpha), & 0 \le \alpha \le 1 \\ 0, & otherwise \end{cases} \quad (39)$$

and

$$\vartheta_{\widetilde{G}(t)}(n(\beta)) = \begin{cases} \beta, n(\beta) = G_1'(\beta), & 0 \le \beta \le 1 \\ \beta, n(\beta) = G_2'(\beta), & 0 \le \beta \le 1 \\ 1, & otherwise \end{cases} \quad (40)$$

where, $0 \le \alpha + \beta \le 1$.

The intuitionistic fuzzy function is denoted as $\widetilde{G}(t)$, and the (α,β)-cut of $\widetilde{G}(t)$ is as follows.

$$[G_1(\alpha), G_2(\alpha); G_1'(\beta), G_2'(\beta)] \quad (41)$$

Example 15. Consider the intuitionistic fuzzy-valued function $\widetilde{F}(t) = \widetilde{a}t^2 + t - 1$.

Due to presence of the intuitionistic coefficient $\widetilde{a}$, the function becomes intuitionistic fuzzy function (see also the graphical representation Figure 8).

The (α,β)-cut of $\widetilde{F}(t)$ is

$$\begin{aligned}[\widetilde{F}(t)]_{\alpha,\beta} = {}& [(3+\alpha^{1/4})t^2 + t - 1, (6 - 2\alpha^{1/4})t^2 + t \\ & - 1; (4 - 2\beta^{1/4})t^2 + t - 1, (4 + \beta^{1/4})t^2 + t - 1]\end{aligned} \quad (42)$$

i.e., $[\widetilde{F}(t)]_{\alpha,\beta} == [(3+\alpha^{1/4}), (6-2\alpha^{1/4}); (4-2\beta^{1/4}), (4+\beta^{1/4})]t^2 + t - 1$.

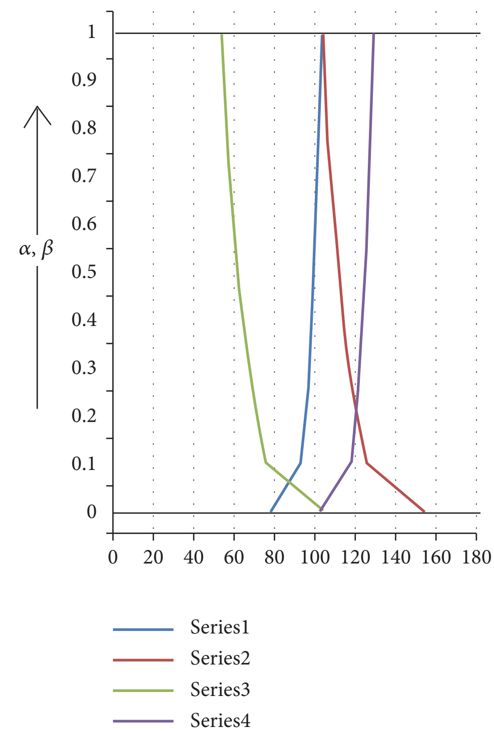

Figure 8: $F_1(\alpha), F_2(\alpha); F_1'(\beta)$ and $F_2'(\beta)$ for $\alpha, \beta \in [0,1]$ at $t = 5$.

3.4. Values and Ambiguities of NIFN. Let A_α^{ni} and A_β^{ni} be any α-cut and β-cut set of a triangular IFN $\widetilde{A}^{ni} = (a_1, a_2, a_3; a_1', a_2, a_3')$, respectively. The values of the membership function $\mu_{\widetilde{A}^{ni}}(x)$ and the nonmembership function $\vartheta_{\widetilde{A}^{ni}}(x)$ for the triangular IFN $\widetilde{A}^{ni}$ are defined as follows:

$$V_\mu(\widetilde{A}^{ni}) = \int_0^1 (L_\alpha(A^{ni}) + R_\alpha(A^{ni})) f(\alpha)\, d\alpha \quad (43)$$

and

$$V_\vartheta(\widetilde{A}^{ni}) = \int_0^1 (L_\beta(A^{ni}) + R_\beta(A^{ni})) g(\beta)\, d\beta. \quad (44)$$

The $f(\alpha)$ is (i) a nonnegative and nondecreasing function on the interval $[0,1]$ and (ii) $f(0) = 0$ and $\int_0^1 f(\alpha)d\alpha = 1/2$. The function $g(\beta)$ is (i) a nonnegative and nonincreasing function on interval $[0,1]$ and (ii) $g(1) = 0$ and $\int_0^1 g(\beta)d\beta = 1/2$.

If we choose $f(\alpha) = \alpha$ and $g(\beta) = 1 - \beta$, then the above assumption is correct.

Here $\alpha \in [0, w]$ and $\beta \in [v, 1]$

The ambiguities of the membership function and the nonmembership function are defined as

$$G_\mu(\widetilde{A}^{ni}) = \int_0^1 (R_\alpha(A^{ni}) - L_\alpha(A^{ni})) f(\alpha)\, d\alpha \quad (45)$$

and

$$G_\vartheta(\widetilde{A}^{ni}) = \int_0^1 (R_\beta(A^{ni}) - L_\beta(A^{ni})) g(\beta)\, d\beta \quad (46)$$

respectively.

Theorem 16. *The values of the membership function and the nonmembership function of nonlinear IFN $\widetilde{A}^{ni}$ are calculated as follows:*

$$V_{\mu}\left(\widetilde{A}^{ni}\right) = \frac{(1-n)\,a_1 + 4na_2 + (1-n)\,a_3}{n+1} \quad (47)$$

and

$$V_{\vartheta}\left(\widetilde{A}^{ni}\right) = \frac{\left(n^2+n+2\right)a_2 + n\left(a_3' + a_1'\right)}{(n+1)(n+2)}. \quad (48)$$

Proof.

$$\begin{aligned} V_{\mu}\left(\widetilde{A}^{ni}\right) &= \int_0^1 \left(L_{\alpha}\left(A^{ni}\right) + R_{\alpha}\left(A^{ni}\right)\right) f(\alpha)\,d\alpha \\ &= \int_0^1 \left(a_1 + \alpha^{1/n}\left(a_2 - a_1\right) + a_3 - \alpha^{1/n}\left(a_3 - a_2\right)\right) \\ &\quad \cdot f(\alpha)\,d\alpha \\ &= \int_0^1 \left(a_1 + a_3 + \alpha^{1/n}\left(2a_2 - a_1 - a_3\right)\right)\alpha\,d\alpha \\ &= \frac{(1-n)\,a_1 + 4na_2 + (1-n)\,a_3}{n+1} \end{aligned} \quad (49)$$

and

$$\begin{aligned} V_{\vartheta}\left(\widetilde{A}^{ni}\right) &= \int_0^1 \left(L_{\beta}\left(A^{ni}\right) + R_{\beta}\left(A^{ni}\right)\right) g(\beta)\,d\beta \\ &= \int_0^1 \left(a_2 - \beta^{1/n}\left(a_2 - a_1'\right) + a_2 + \beta^{1/n}\left(a_3' - a_2\right)\right) \\ &\quad \cdot g(\beta)\,d\beta = \int_0^1 \left(2a_2 + \beta^{1/n}\left(a_3' + a_1' - 2a_2\right)\right) \\ &\quad \cdot (1-\beta)\,d\beta = \frac{\left(n^2+n+2\right)a_2 + n\left(a_3' + a_1'\right)}{(n+1)(n+2)}. \end{aligned} \quad (50)$$

□

Theorem 17. *The ambiguities of the membership function and the nonmembership function of nonlinear IFN $\widetilde{A}^{ni}$ are calculated as follows:*

$$G_{\mu}\left(\widetilde{A}^{ni}\right) = \frac{\left(a_3 - a_1\right)}{2(2n+1)} \quad (51)$$

and

$$G_{\vartheta}\left(\widetilde{A}^{ni}\right) = \frac{n^2\left(a_3' - a_1'\right)}{(n+1)(2n+1)} \quad (52)$$

Proof.

$$\begin{aligned} G_{\mu}\left(\widetilde{A}^{ni}\right) &= \int_0^1 \left(R_{\alpha}\left(A^{ni}\right) - L_{\alpha}\left(A^{ni}\right)\right) f(\alpha)\,d\alpha \\ &= \int_0^1 \left(a_3 - \alpha^{1/n}\left(a_3 - a_2\right) - a_1 - \alpha^{1/n}\left(a_2 - a_1\right)\right) \\ &\quad \cdot f(\alpha)\,d\alpha = \int_0^1 \left(a_3 - a_1 - \alpha^{1/n}\left(a_3 - a_1\right)\right)\alpha\,d\alpha \\ &= \int_0^1 \left(a_3 - a_1\right)\left(1 - \alpha^{1/n}\right)\alpha\,d\alpha = \frac{\left(a_3 - a_1\right)}{2(2n+1)}. \end{aligned} \quad (53)$$

and

$$\begin{aligned} G_{\vartheta}\left(\widetilde{A}^{ni}\right) &= \int_0^1 \left(R_{\beta}\left(A^{ni}\right) - L_{\beta}\left(A^{ni}\right)\right) g(\beta)\,d\beta \\ &= \int_0^1 \left(a_2 + \beta^{1/n}\left(a_3' - a_2\right) - a_2 + \beta^{1/n}\left(a_2 - a_1'\right)\right) \\ &\quad \cdot g(\beta)\,d\beta = \int_0^1 \left(a_3' - a_1'\right)\beta^{1/n}(1-\beta)\,d\beta \\ &= \frac{n^2\left(a_3' - a_1'\right)}{(n+1)(2n+1)}. \end{aligned} \quad (54)$$

□

3.5. Ranking of Intuitionistic Fuzzy Number Using Valuation and Ambiguity. If we wish to find the ranking of an intuitionistic fuzzy number, then we need to define valuation and ambiguity index.

The valuation index is denoted as follows

$$V\left(\widetilde{A}^{ni}, \lambda\right) = V_{\mu}\left(\widetilde{A}^{ni}\right) + \lambda\left(V_{\mu}\left(\widetilde{A}^{ni}\right) - V_{\vartheta}\left(\widetilde{A}^{ni}\right)\right) \quad (55)$$

and the ambiguity index is denoted as follows

$$A\left(\widetilde{A}^{ni}, \lambda\right) = G_{\vartheta}\left(\widetilde{A}^{ni}\right) - \lambda\left(G_{\vartheta}\left(\widetilde{A}^{ni}\right) - G_{\mu}\left(\widetilde{A}^{ni}\right)\right) \quad (56)$$

for $\lambda \in [0,1]$.

Now λ can be chosen by different ways. It depends upon decision making choice.

The range of λ, $\lambda \in [0, 1/2)$ indicates the decision maker's pessimistic attitude and $\lambda \in (1/2, 1]$ indicates the decision maker's optimistic attitude towards uncertainty.

Generally it is better to take $\lambda = 1/2$.

The ranking of the given number can be written as follows.

$$R\left(\widetilde{A}^{ni}\right) = V\left(\widetilde{A}^{ni}, \frac{1}{2}\right) + A\left(\widetilde{A}^{ni}, \frac{1}{2}\right) \quad (57)$$

Basically we cannot compare between two intuitionistic fuzzy numbers, that is, determining which one is greater than, equal to, or less than the other. But using the concept of ranking we can give some approximate relation between them. There exist different literature sources where researchers define ranking in their way. For different definition the comparison may be different.

For our definition we can conclude that

(1) If $R(\widetilde{A}^{ni}) \gtrsim R(\widetilde{B}^{ni})$ then $\widetilde{A}^{ni} \gtrsim \widetilde{B}^{ni}$

(2) If $R(\widetilde{A}^{ni}) \cong R(\widetilde{B}^{ni})$ then $\widetilde{A}^{ni} \cong \widetilde{B}^{ni}$

(3) If $R(\widetilde{A}^{ni}) \lesssim R(\widetilde{B}^{ni})$ then $\widetilde{A}^{ni} \lesssim \widetilde{B}^{ni}$

Table 2

Comparison of two intuitionistic fuzzy numbers		
	Valuation of the membership function and the nonmembership function	
$V_\mu(\widetilde{A}^{ni}) = 49.66$		$V_\mu(\widetilde{B}^{ni}) = 49$
$V_\vartheta(\widetilde{A}^{ni}) = 24.86$		$V_\vartheta(\widetilde{B}^{ni}) = 24.73$
	Ambiguities of membership function and nonmembership function	
$G_\mu(\widetilde{A}^{ni}) = 1.12$		$G_\mu(\widetilde{B}^{ni}) = 1.12$
$G_\vartheta(\widetilde{A}^{ni}) = 0.58$		$G_\vartheta(\widetilde{B}^{ni}) = 0.66$
	Valuation index	
$V(\widetilde{A}^{ni}, \lambda) = 62.06$		$V(\widetilde{B}^{ni}, \lambda) = 61.13$
	Ambiguity index	
$A(\widetilde{A}^{ni}, \lambda) = 0.85$		$A(\widetilde{B}^{ni}, \lambda) = 0.89$
	Ranking	
$R(\widetilde{A}^{ni}) = 62.91$		$R(\widetilde{B}^{ni}) = 62.02$

Where the symbol "$\widetilde{>}$", "$\widetilde{=}$" and "$\widetilde{<}$" defines the greater than, equal to, and less than relation in fuzzy sense.

Example 18. Let $\widetilde{A}^{ni} = (22, 25, 27; 21, 25, 28)$ and $\widetilde{B}^{ni} = (21, 25, 26; 20, 25, 28)$, $n = 1/2$; then compare the numbers.

Solution. See Table 2.

Hence $\widetilde{A}^{ni} \widetilde{>} \widetilde{B}^{ni}$, i.e., $\widetilde{A}^{ni}$ is greater than $\widetilde{B}^{ni}$ in fuzzy sense.

4. De-i-Fuzzification Based on Average of (α,β)-Cut Method

The crispification value of an intuitionistic fuzzy number is named as de-i-fuzzification value [33]. Here we tried to find the de-i-fuzzification value of NTIFN using average of (α, β)-cut method.

4.1. De-i-Fuzzification Based on Average of (α, β)-Cut Method of an IFN. For an IFN $\widetilde{A}^{i}$, the de-i-fuzzification value of $\widetilde{A}^{i}$ is a crisp value which can be derived in the following way

$$\widehat{A}^{i} = \frac{\widehat{A}^{i}_{\alpha} + \widehat{A}^{i}_{\beta}}{2} \tag{58}$$

where $\widehat{A}^{i}_{\alpha}$ is de-i-fuzzification of α-cut and $\widehat{A}^{i}_{\beta}$ is de-i-fuzzification value of β-cut.

That is, $\widehat{A}^{i}_{\alpha} = \int_0^1 ((A_1(\alpha) + A_2(\alpha))/2) d\alpha$ and $\widehat{A}^{i}_{\beta} = \int_0^1 ((A_1'(\beta) + A_2'(\beta))/2) d\beta$.

Now if $\widetilde{A}^{i} = (a_1, a_2, a_3; a_1', a_2, a_3')$ is intuitionistic triangular fuzzy number, then its defuzzification value is as follows.

$$\widehat{A}^{i} = \frac{a_1 + a_1' + 4a_2 + a_3 + a_3'}{8} \tag{59}$$

4.2. De-i-Fuzzification Method For NTIFN. We can find the de-i-fuzzification value of the NTIFN $\widetilde{A}^{ni}_{TIFN} = (a_1, a_2, a_3; a_1', a_2, a_3')$ by the above said method. If $p = q = r = s = n$, then the de-i-fuzzification value is defined as follows.

$$\widehat{A}^{ni} = \frac{na_1 + na_3 + 2(n+1)a_2 + na_1' + na_3'}{4(n+1)} \tag{60}$$

Example 19. Find the de-i-fuzzified value of a nonlinear intuitionistic fuzzy number $\widetilde{A}^{ni}_{TIFN} = (105, 110, 116; 104, 110, 115)$. Here $p = q = r = s = 2$.

Solution. The de-i-fuzzified value of the number is $\widehat{A}^{ni} = 91.66$.

5. Intuitionistic Fuzzy Distance and Integrals

5.1. Generalized Hukuhara Distance on Intuitionistic Fuzzy-Valued Function

Definition 20. The Hausdorff distance between intuitionistic fuzzy numbers is given by $D : \mathfrak{R}_{\mathscr{F}} \times \mathfrak{R}_{\mathscr{F}} \longrightarrow \mathbb{R}^{+} \cup \{0\}$ as in

$$\begin{aligned} D(u, v; p, q) &= \sup_{\alpha,\beta\in[0,1]} d\left([u]_\alpha, [v]_\alpha; [p]_\beta, [q]_\beta\right) \\ &= \sup_{\alpha,\beta\in[0,1]} \max\{|u_1(\alpha) - v_1(\alpha)|, |u_2(\alpha) - v_2(\alpha)|, \\ &\quad |p_1(\beta) - q_1(\beta)|, |p_2(\beta) - q_2(\beta)|\} \end{aligned} \tag{61}$$

where d is Hausdorff metric and metric space $(\mathfrak{R}_{\mathscr{F}}, D)$ is complete, separable, and locally compact. The following substances for metric D are tenable:

(1) $D(u \oplus w, v \oplus w; p \oplus z, q \oplus z) = D(u, v; p, q), \forall u, v, p, q, w, z \in \mathfrak{R}_{\mathscr{F}}$;

(2) $D(\lambda u, \lambda v; \lambda p, \lambda q) = |\lambda| D(u, v; p, q), \lambda \in \mathbb{R}, u, v, p, q \in \mathfrak{R}_{\mathscr{F}}$;

(3) $D(u_1 \oplus u_2, v_1 \oplus v_2; p_1 \oplus p_2, q_1 \oplus q_1) \leq D(u_1, v_1; p_1, q_1) + D(u_2, v_2; p_2, q_1)\ \forall u_1, u_2, v_1, v_2, p_1, p_2, q_1, q_1 \in \mathfrak{R}_{\mathscr{F}}$;

(4) $D(u_1 \ominus u_2, v_1 \ominus v_2; p_1 \ominus p_2, q_1 \ominus q_1) \leq D(u_1, v_1; p_1, q_1) + D(u_2, v_2; p_2, q_1)$, as long as $u_1 \ominus u_2, v_1 \ominus v_2; p_1 \ominus p_2, q_1 \ominus q_1$ exists and $u_1, u_2, v_1, v_2, p_1, p_2, q_1, q_1 \in \mathfrak{R}_{\mathcal{F}}$.

Lemma 21. *Let $u_n, v_n, p_n, q_n, u, v, p, q \in \mathfrak{R}_{\mathcal{F}}, n \in \mathbb{N}$. Let $u_n \longrightarrow u, v_n \longrightarrow v, p_n \longrightarrow p, q_n \longrightarrow q$, as $n \longrightarrow \infty$. Then $D(u_n, v_n; p_n, q_n) \longrightarrow D(u, v; p, q)$, as $n \longrightarrow \infty$. In particular we can write the following.*

$$\begin{aligned} &\lim_{n\longrightarrow+\infty} D(u_n, v_n; p_n, q_n) \\ &= D\left(\lim_{n\longrightarrow+\infty} u_n, \lim_{n\longrightarrow+\infty} v_n; \lim_{n\longrightarrow+\infty} p_n, \lim_{n\longrightarrow+\infty} q_n\right) \\ &= D(u, v; p, q) \end{aligned} \quad (62)$$

Lemma 22. *Let $u_n, u \in \mathfrak{R}_{\mathcal{F}}$ and $c_n, c \in \mathbb{R}$, such that $u_n \longrightarrow u$ and $c_n \longrightarrow c$ as $n \longrightarrow +\infty$. Therefore in D-metric*

$$\begin{aligned} \lim_{n\longrightarrow+\infty} \left(u_n \odot c_n\right) &= \left(\lim_{n\longrightarrow+\infty} u_n\right) \odot \left(\lim_{n\longrightarrow+\infty} c_n\right) \\ &= u \odot c. \end{aligned} \quad (63)$$

Definition 23. The generalized Hukuhara difference between two intuitionistic fuzzy numbers $a, b \in \mathfrak{R}_{\mathcal{F}}$ is defined as follows.

$$a \ominus_{gH} b = k \Longleftrightarrow \begin{cases} a = b + k \\ b = a + (-1)k \end{cases} \quad (64)$$

The (α, β)-cut set

$$\begin{aligned} \left[a \ominus_{gH} b\right]_{\alpha,\beta} = &\left[\min\{r_1(\alpha), r_2(\alpha)\},\right. \\ &\max\{r_1(\alpha), r_2(\alpha)\}; \min\{s_1(\beta), s_2(\beta)\}, \\ &\left.\max\{s_1(\beta), s_2(\beta)\}\right] \end{aligned} \quad (65)$$

where $r_1(\alpha) = a_1(\alpha) - b_2(\alpha)$, $r_2(\alpha) = a_2(\alpha) - b_1(\alpha)$, $s_1(\beta) = a_1'(\beta) - b_2'(\beta)$ and $s_2(\beta) = a_2'(\beta) - b_1'(\beta)$.

Let $e = a \ominus_{gH} b$.

The existence conditions for which $a \ominus_{gH} b$ exists are as follows:

(1) $e_1(\alpha) = a_1(\alpha) - b_2(\alpha)$, $e_2(\alpha) = a_2(\alpha) - b_1(\alpha)$, $e_1'(\beta) = a_1'(\beta) - b_2'(\beta)$, and $e_2'(\beta) = a_2'(\beta) - b_1'(\beta)$, with (i) $e_1(\alpha)$, $e_2'(\beta)$ being increasing and $e_2(\alpha)$, $e_1'(\beta)$ being decreasing function and (ii) $e_1(\alpha) \leq e_2(\alpha)$, $e_2'(\beta) \leq e_1'(\beta)$ for all $\alpha, \beta \in [0, 1]$.

(2) $e_2(\alpha) = a_1(\alpha) - b_2(\alpha)$, $e_1(\alpha) = a_2(\alpha) - b_1(\alpha)$, $e_2'(\beta) = a_1'(\beta) - b_2'(\beta)$, and $e_1'(\beta) = a_2'(\beta) - b_1'(\beta)$, with (i) $e_1(\alpha)$, $e_2'(\beta)$ being increasing and $e_2(\alpha)$, $e_1'(\beta)$ being decreasing function and (ii) $e_1(\alpha) \leq e_2(\alpha)$, $e_2'(\beta) \leq e_1'(\beta)$ for all $\alpha, \beta \in [0, 1]$.

Remark 24. In the whole paper, we considered $a \ominus_{gH} b \in \mathfrak{R}_{\mathcal{F}}$.

6. Linear Fredholm Integral Equation in Intuitionistic Fuzzy Environment

Integral equations are very important in the area of calculus theory. They appear in different application forms. Practically, when we modeled with integral equation the uncertainty parameters can arise. For that purpose we need to study imprecise integral equation. In this paper we study the intuitionistic integral equation.

6.1. Intuitionistic Fuzzy Integral Equation. Considering the linear Fredholm integral equation of second kind

$$u(x) = f(x) + \lambda \int_a^b k(x, t)\, u(t)\, dt \quad (66)$$

where $x \in D$, $u(x)$, and $f(x)$ are functions on $D = [a, b]$, $k(x, t)$ is arbitrary kernel function over $\mathcal{T} = [a, b] \times [a, b]$, and u is unknown on D.

The upstairs integral equation is said to be intuitionistic integral equation if

(1) $f(x)$ is intuitionistic fuzzy-valued function;

(2) only $k(x, t)$ is intuitionistic fuzzy-valued function;

(3) both $f(x)$ and $k(x, t)$ are intuitionistic fuzzy-valued functions.

6.2. Condition for Existence for Solution Intuitionistic Fuzzy Integral Equation. Consider the following intuitionistic fuzzy integral equation.

$$u(x) = f(x) + \lambda \int_a^b k(x, t)\, u(t)\, dt \quad (67)$$

Let the solution of the above IFIE be $\tilde{u}(x)$ and its (α, β)-cut be $u(x)(\alpha, \beta) = [u_1(x, \alpha), u_2(x, \alpha); u_1'(x, \beta), u_2'(x, \beta)]$.

The solution is called strong solution if

(i) $\partial u_1(x, \alpha)/\partial\alpha > 0$, $\partial u_2(x, \alpha)/\partial\alpha < 0$ $\forall \alpha \in [0, 1]$, $u_1(x, 1) \leq u_2(x, 1)$

and

(ii) $\partial u_1'(x, \beta)/\partial\beta < 0$, $\partial u_2'(x, \beta)/\partial\beta > 0$ $\forall \beta \in [0, 1]$, $u_1'(x, 0) \leq u_2'(x, 0)$.

In the rest of the cases, the solution is a weak solution.

6.3. Solution of Intuitionistic Fuzzy Integral Equation

Problem 25. Consider the integral equation $u(x) = \tilde{f}(x) + \lambda \int_a^b \tilde{k}(x, t) u(t) dt$.

(In this integral equation $\tilde{f}(x)$ and $\tilde{k}(x, t)$ are intuitionistic fuzzy function.)

Solution. The two possible cases are as follows.

Case 1 (when $\tilde{k}(x, t)$ is positive function). In this case if we take fuzzy Laplace transformation, then the intuitionistic fuzzy integral equation becomes

$$\boldsymbol{L}[u(x)] = \boldsymbol{L}\left[\tilde{f}(x)\right] + \lambda \boldsymbol{L}\left[\tilde{k}(x, t)\right] \boldsymbol{L}[u(t)] \quad (68)$$

which implies the following results.

$$u_1(x, \alpha) = l^{-1}\left\{\frac{l\{f_1(s, \alpha)\}}{1 - \lambda l\{k_1(s, \alpha)\}}\right\}$$

$$u_2(x,\alpha) = l^{-1}\left\{\frac{l\{f_2(s,\alpha)\}}{1-\lambda l\{k_2(s,\alpha)\}}\right\}$$
$$u_1'(x,\beta) = l^{-1}\left\{\frac{l\{f_1'(s,\beta)\}}{1-\lambda l\{k_1'(s,\alpha)\}}\right\}$$
$$u_2'(x,\beta) = l^{-1}\left\{\frac{l\{f_2'(s,\beta)\}}{1-\lambda l\{k_2'(s,\beta)\}}\right\} \quad (69)$$

Case 2 (when $\tilde{k}(x,t)$ is negative function). Let $\tilde{k}(x,t) = -\widetilde{m}(x,t)$.

In this case if we take fuzzy Laplace transformation, then the intuitionistic fuzzy integral equation becomes

$$\boldsymbol{L}[u(x)] = \boldsymbol{L}\left[\tilde{f}(x)\right] + \lambda\boldsymbol{L}[-\widetilde{m}(x,t)]\,\boldsymbol{L}[u(t)] \quad (70)$$

which implies the following results.

$$u_1(x,\alpha) = l^{-1}\left\{\frac{\lambda l\{m_1(x,t,\alpha)\}\,l\{f_1(s,\alpha)\}}{1-\lambda^2 l\{m_1(x,t,\alpha)\}\,l\{m_2(x,t,\alpha)\}}\right\}$$
$$u_2(x,\alpha) = l^{-1}\left\{\frac{\lambda l\{m_2(x,t,\alpha)\}\,l\{f_2(s,\alpha)\}}{1-\lambda^2 l\{m_1(x,t,\alpha)\}\,l\{m_2(x,t,\alpha)\}}\right\}$$
$$u_1'(x,\beta) = l^{-1}\left\{\frac{\lambda l\{m_1'(x,t,\beta)\}\,l\{f_1'(s,\beta)\}}{1-\lambda^2 l\{m_1'(x,t,\beta)\}\,l\{m_2'(x,t,\beta)\}}\right\} \quad (71)$$
$$u_2'(x,\beta) = l^{-1}\left\{\frac{\lambda l\{m_2'(x,t,\beta)\}\,l\{f_2'(s,\beta)\}}{1-\lambda^2 l\{m_1'(x,t,\beta)\}\,l\{m_2'(x,t,\beta)\}}\right\}$$

Example 26. Consider the integral equation $u(x) = \tilde{f}(x) + \lambda\int_a^b k(x,t)u(t)dt$, where $\tilde{f}(x)$ is a intuitionistic fuzzy-valued function defined as $\tilde{f}(x) = (1,2,2.5;0.5,2,3)e^{-x}$ and $\lambda = 1$, $a = 0, b = x, k(x,t) = \sin(x-t)$ (here $p = 2, q = 3, r = 4$ and $s = 2$).

Solution. Taking intuitionistic fuzzy Laplace transform we can find the solution as follows.

$$u_1(x,\alpha) = \left(1+\alpha^{1/2}\right)\left(2e^{-x}+x-1\right)$$
$$u_2(x,\alpha) = \left(2.5-0.5\alpha^{1/3}\right)\left(2e^{-x}+x-1\right)$$
$$u_1'(x,\beta) = \left(2-1.5\beta^{1/4}\right)\left(2e^{-x}+x-1\right) \quad (72)$$
$$u_2'(x,\beta) = \left(2+\beta^{1/2}\right)\left(2e^{-x}+x-1\right)$$

Remark. Clearly from Figure 9 we see that $u_1(x,\alpha)$, $u_2'(x,\beta)$ is increasing function and $u_2(x,\alpha)$, $u_1'(x,\beta)$ is decreasing function at $x = 2$. Hence at this particular point the solution is a strong solution.

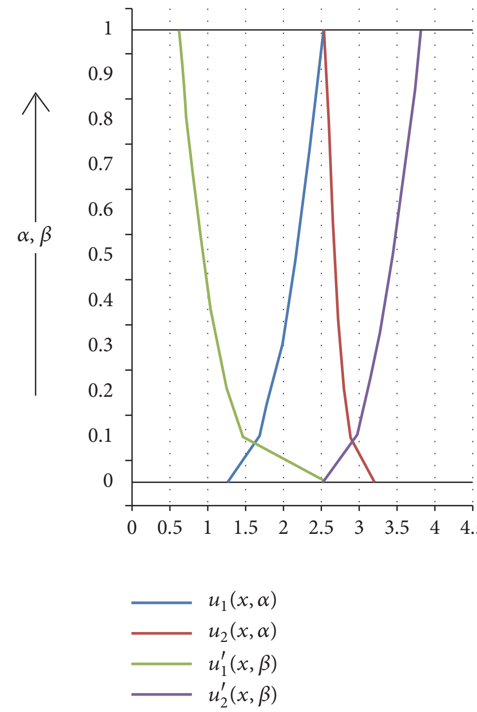

Figure 9: Plot of $u_1(x,\alpha)$, $u_2(x,\alpha)$, $u_1'(x,\beta)$, and $u_2'(x,\beta)$ at $x = 2$ for $\alpha, \beta \in [0,1]$.

Defuzzification Value. The defuzzification value of solution is given by the following.

$$\hat{u}(x) = 1.81\left(2e^{-x}+x-1\right) \quad (73)$$

7. Conclusion

It is not necessary that the membership and the nonmembership function of an intuitionistic fuzzy number be linear. They may be nonlinear. If they are nonlinear, then their order may be fraction or may not. For taking the above concept we introduce nonlinear intuitionistic fuzzy numbers (NIFN) and their arithmetic operation. We apply max-min principle for arithmetic operation on NIFN. The exact resulting number is written in parametric form. Finally we use this number in integral equation in intuitionistic fuzzy environment. For helping the readers who try to compare the fuzzy solution with crisp number, we defuzzify or crispify the result by min of (α,β)-cut method. The paper can help those who are modeled with uncertainty and integral and differential calculus. In future any one can use this concept for finding various types of fuzzy numbers and apply them in various fields of mathematics.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The first author of the article wishes to convey his heartiest thanks to Miss. Gullu for inspiring him to write the article.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Computation*, vol. 8, pp. 338–353, 1965.

[2] S. S. L. Chang and L. A. Zadeh, "On fuzzy mapping and control," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 2, pp. 30–34, 1972.

[3] P. Diamond and P. Kloeden, *Metric Spaces of Fuzzy Sets: Theory and Applications*, World Scientific, Singapore, 1994.

[4] D. Dubois and H. Prade, "Operations on fuzzy numbers," *International Journal of Systems Science*, vol. 9, no. 6, pp. 613–626, 1978.

[5] J. Goetschel and W. Voxman, "Elementary fuzzy calculus," *Fuzzy Sets and Systems*, vol. 18, no. 1, pp. 31–43, 1986.

[6] O. Kaleva, "Fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 24, no. 3, pp. 301–317, 1987.

[7] K. T. Atanassov, "Intuitionistic fuzzy sets," in *VII ITKR's Session*, Sofia, Bulgarian, 1983.

[8] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[9] A. I. Ban, *Intuitionistic Fuzzy Measures. Theory and Applications*, Nova Science, New York, NY, USA, 2006.

[10] E. Szmidt and J. Kacprzyk, "Distances between intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 114, no. 3, pp. 505–518, 2000.

[11] A. Ahmadian, F. Ismail, S. Salahshour, D. Baleanu, and F. Ghaemi, "Uncertain viscoelastic models with fractional order: a new spectral tau method to study the numerical simulations of the solution," *Communications in Nonlinear Science and Numerical Simulation*, vol. 53, pp. 44–64, 2017.

[12] S. Salahshour, A. Ahmadian, F. Ismail, D. Baleanu, and N. Senu, "A New fractional derivative for differential equation of fractional order under interval uncertainty," *Advances in Mechanical Engineering*, vol. 7, no. 12, Article ID 15619138, 2015.

[13] A. Ahmadian, S. Salahshour, C. S. Chan, and D. Baleanu, "Numerical solutions of fuzzy differential equations by an efficient Runge-Kutta method with generalized differentiability," *Fuzzy Sets and Systems*, vol. 331, pp. 47–67, 2018.

[14] A. Ahmadian, S. Salahshour, and C. S. Chan, "A Runge–Kutta method with reduced number of function evaluations to solve hybrid fuzzy differential equations," *Soft Computing*, vol. 19, no. 4, pp. 1051–1062, 2015.

[15] A. Ahmadian, H. S. Chan, S. Salahshour, and V. Vaitheeswaran, "FTFBE: A numerical approximation for fuzzy time-fractional Bloch equation," in *Proceedings of the 2014 IEEE International Conference on Fuzzy Systems, FUZZ-IEEE 2014*, pp. 418–423, China, July 2014.

[16] A. Ahmadian, S. Salahshour, and C. S. Chan, "Fractional differential systems: a fuzzy solution based on operational matrix of shifted chebyshev polynomials and its applications," *IEEE Transactions on Fuzzy Systems*, vol. 25, no. 1, pp. 218–236, 2017.

[17] K. Atanassov, P. Vassilev, and R. Tsvetkov, *Intuitionistic Fuzzy Sets, Measures and Integrals*, Prof. M. Drinov" Academic Publishing House, Sofia, Bulgaria, 2013, 2013 is devoted to intuitionistic fuzzy integrals.

[18] M. Friedman, M. Ming, and A. Kandel, "On fuzzy integral equations," *Fundamenta Informaticae*, vol. 37, no. 1-2, pp. 89–99, 1999.

[19] H. Hosseini Fadravi, R. Buzhabadi, and H. Saberi Nik, "Solving linear Fredholm fuzzy integral equations of the second kind by artificial neural networks," *Alexandria Engineering Journal*, vol. 53, no. 1, pp. 249–257, 2014.

[20] M. Ghanbari and T. Allahviranloo, "A new application of homotopy analysis method," *Thai Journal of Mathematics*, vol. 10, no. 1, pp. 43–57, 2012.

[21] T. Allahviranloo and S. Hashemzehi, "The homotopy perturbation method for fuzzy fredholm integral equations," *Journal of Applied Mathematics, Islamic Azad University of Lahijan*, vol. 5, no. 19, 12 pages, 2008.

[22] M. Baghmisheh and R. Ezzati, "Numerical solution of nonlinear fuzzy Fredholm integral equations of the second kind using hybrid of block-pulse functions and Taylor series," *Advances in Difference Equations*, vol. 2015, 15 pages, 2015.

[23] M. Jahantigh, T. Allahviranloo, and M. Otadi, "Numerical solution of fuzzy integral equations," *Applied Mathematical Sciences*, vol. 2, no. 1-4, pp. 33–46, 2008.

[24] S. Qasim Hasan and A. J. Abdulqader, "Numerical and analytic method for solving proposal fuzzy nonlinear volterra integral equation by using homotopy analysis method," *American Journal of Applied Mathematics*, vol. 4, no. 3, pp. 142–157, 2016.

[25] E. A. Hussain and A. W. Ali, "Linear Volterra fuzzy integral equation solved by modified trapezoidal method," *International Journal of Applied Mathematics & Statistical Sciences*, vol. 2, pp. 43–54, 2013.

[26] S. Narayanamoorthy and S. P. Sathiyapriya, "Homotopy perturbation method: a versatile tool to evaluate linear and nonlinear fuzzy Volterra integral equations of the second kind," *SpringerPlus*, vol. 5, no. 1, 16 pages, 2016.

[27] A. Jafarian and M. S. Measoomy Nia, "Application of Taylor expansion method for the Volterra fuzzy integral equations system," *Acta Universitatis Matthiae Belii, Series Mathematics*, vol. 20, pp. 18–31, 2012.

[28] P. Diamond, "Theory and applications of fuzzy volterra integral equations," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 1, pp. 97–102, 2002.

[29] P. Ravi, D. O'Regan, and V. Lakshmikantham, "Fuzzy Volterra Integral Equations: A Stacking Theorem Approach," *Applicable Analysis: An International Journal*, vol. 83, no. 5, pp. 521–532, 2004.

[30] S. P. Mondal and T. K. Roy, "First order non homogeneous ordinary differential equation with initial value as triangular intuitionistic fuzzy number," *Journal of Uncertain Systems*, vol. 9, no. 4, pp. 274–285, 2015.

[31] S. P. Mondal and T. K. Roy, "Generalized intuitionistic fuzzy Laplace transform and its application in electrical circuit," *TWMS Journal of Applied and Engineering Mathematics*, vol. 5, no. 1, pp. 30–45, 2015.

[32] S. P. Mondal, "Non-linear intuitionistic fuzzy number and its application in partial differential equation," *Emerging Research on Applied Fuzzy Sets and Intuitionistic Fuzzy Matrices*, pp. 215–234, 2016.

[33] A. Ban, J. Kacprzyk, and K. T. Atanassov, "On de-I-fuzzification of intuitionistic fuzzy sets," *Comptes Rendus de l'Academie bulgare des Sciences*, vol. 61, no. 12, pp. 1535–1540, 2008.