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Research Article

An Analytical Approach to Evaluating Nonmonotonic Functions of Fuzzy Numbers

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This paper presents a novel analytical approach to evaluating continuous, nonmonotonic functions of independent fuzzy numbers. The approach is based on a parametric α-cut representation of fuzzy numbers and allows for the inclusion of parameter uncertainties into mathematical models.

1. Introduction

This paper continues the research from our previous work [1]. In [1], we formulated a practical analytical approach to evaluating continuous, monotonic functions of independent fuzzy numbers, which is based on an alternative formulation of the extension principle [2]. In this paper, we extend this approach to general, nonmonotonic functions of independent fuzzy numbers. For the theoretical background, we assume the reader is familiar with Sections 2 and 4 from [1].

2. Preliminaries

In the remainder of this paper, we will use two typical fuzzy numbers in engineering, the triangular and the Gaussian fuzzy number. The definitions are provided below.

Definition 1. The *triangular fuzzy number* (TFN) [3]

$$\tilde{x} = \operatorname{tfn}\left(\overline{x}, \tau^{\mathrm{L}}, \tau^{\mathrm{R}}\right) \tag{1}$$

is defined by the membership function

$$\mu_{\tilde{x}}(x) = \begin{cases} 1 + \dfrac{x - \overline{x}}{\tau^{\mathrm{L}}}, & \overline{x} - \tau^{\mathrm{L}} \le x \le \overline{x}, \\ 1 - \dfrac{x - \overline{x}}{\tau^{\mathrm{R}}}, & \overline{x} < x \le \overline{x} + \tau^{\mathrm{R}}, \end{cases} \tag{2}$$

where $\overline{x}$ denotes the *modal value*, τ^{L} denotes the *left-hand*, and τ^{R} denotes the *right-hand spread* of $\tilde{x}$. If $\tau^{\mathrm{L}} = \tau^{\mathrm{R}}$, the TFN is called *symmetric*. Its α-cuts $x(\alpha) = [x^{\mathrm{L}}(\alpha), x^{\mathrm{R}}(\alpha)]$ result from the inverse functions of (2) with respect to x:

$$\begin{aligned} x^{\mathrm{L}}(\alpha) &= \overline{x} - \tau^{\mathrm{L}}(1 - \alpha), \quad 0 < \alpha \le 1, \\ x^{\mathrm{R}}(\alpha) &= \overline{x} + \tau^{\mathrm{R}}(1 - \alpha), \quad 0 < \alpha \le 1. \end{aligned} \tag{3}$$

Definition 2. The *Gaussian fuzzy number* (GFN) [3]

$$\tilde{x} = \operatorname{gfn}\left(\overline{x}, \sigma^{\mathrm{L}}, \sigma^{\mathrm{R}}\right) \tag{4}$$

is defined by the membership function

$$\mu_{\tilde{x}}(x) = \begin{cases} \exp\left[-\dfrac{1}{2}\left(\dfrac{x - \overline{x}}{\sigma^{\mathrm{L}}}\right)^2\right], & x \le \overline{x}, \\ \exp\left[-\dfrac{1}{2}\left(\dfrac{x - \overline{x}}{\sigma^{\mathrm{R}}}\right)^2\right], & x > \overline{x}, \end{cases} \tag{5}$$

where $\overline{x}$ denotes the *modal value*, σ^{L} denotes the *left-hand*, and σ^{R} denotes the *right-hand standard deviation* of $\tilde{x}$. If $\sigma^{\mathrm{L}} = \sigma^{\mathrm{R}}$, the GFN is called *symmetric*. Its α-cuts $x(\alpha) = [x^{\mathrm{L}}(\alpha), x^{\mathrm{R}}(\alpha)]$ result in

$$\begin{aligned} x^{\mathrm{L}}(\alpha) &= \overline{x} - \sigma^{\mathrm{L}}\sqrt{-2\ln(\alpha)}, \quad 0 < \alpha \le 1, \\ x^{\mathrm{R}}(\alpha) &= \overline{x} + \sigma^{\mathrm{R}}\sqrt{-2\ln(\alpha)}, \quad 0 < \alpha \le 1. \end{aligned} \tag{6}$$

3. Analytical Approach

For evaluating continuous, nonmonotonic functions of independent fuzzy numbers, the authors in [4, 5] suggest including the extreme points as constant profiles into the computation. However, this is not enough and can lead to erroneous results, as was pointed out in [6]. More specifically, all permutations of the interval boundaries of $x_m(\alpha)$, $m = 1, \ldots, n$, with the components of the extreme points have to be considered as well.

Basically, our analytical approach can be classified into two parts depending on the monotonicity of f: a general and an extended part.

3.1. General Part. If the function f is nonmonotonic in all x_m, $m = 1, \ldots, n$, we can obtain the analytical solution as follows.

(1) Evaluate the function f for all the 2^n permutations of the interval boundaries of $x_m(\alpha) = [x_m^{\mathrm{L}}(\alpha), x_m^{\mathrm{R}}(\alpha)]$, $m = 1, \ldots, n$. For example, if $n = 2$, then compute

$$\begin{aligned} y^{\mathrm{LL}}(\alpha) &= f\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{L}}(\alpha)\right), \\ y^{\mathrm{LR}}(\alpha) &= f\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{R}}(\alpha)\right), \\ y^{\mathrm{RL}}(\alpha) &= f\left(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{L}}(\alpha)\right), \\ y^{\mathrm{RR}}(\alpha) &= f\left(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{R}}(\alpha)\right). \end{aligned} \tag{7}$$

(2) Evaluate the function f for all the $2ns$ combinations of the interval boundaries of $x_m(\alpha) = [x_m^{\mathrm{L}}(\alpha), x_m^{\mathrm{R}}(\alpha)]$, $m = 1, \ldots, n$, with the components of the extreme points $(x_{1,r}^*, \ldots, x_{n,r}^*)$, $r = 1, \ldots, s$. For example, if $n = 3$ and $s = 1$, then compute

$$\begin{aligned} y^{\mathrm{L11}}(\alpha) &= f\left(x_1^{\mathrm{L}}(\alpha), x_{2,1}^*, x_{3,1}^*\right), \\ y^{\mathrm{R11}}(\alpha) &= f\left(x_1^{\mathrm{R}}(\alpha), x_{2,1}^*, x_{3,1}^*\right), \\ y^{\mathrm{1L1}}(\alpha) &= f\left(x_{1,1}^*, x_2^{\mathrm{L}}(\alpha), x_{3,1}^*\right), \\ y^{\mathrm{1R1}}(\alpha) &= f\left(x_{1,1}^*, x_2^{\mathrm{R}}(\alpha), x_{3,1}^*\right), \\ y^{\mathrm{11L}}(\alpha) &= f\left(x_{1,1}^*, x_{2,1}^*, x_3^{\mathrm{L}}(\alpha)\right), \\ y^{\mathrm{11R}}(\alpha) &= f\left(x_{1,1}^*, x_{2,1}^*, x_3^{\mathrm{R}}(\alpha)\right). \end{aligned} \tag{8}$$

(3) If $(x_{1,r}^*, \ldots, x_{n,r}^*) \in \operatorname{supp}(\widetilde{x}_1) \times \cdots \times \operatorname{supp}(\widetilde{x}_n)$ for certain $r \in \{1, \ldots, s\}$, compute the corresponding $y^{*r} = f(x_{1,r}^*, \ldots, x_{n,r}^*)$.

(4) Plot all solution candidates in the same diagram.

(5) Finally, starting from the modal point $\overline{y} = f(\overline{x}_1, \ldots, \overline{x}_n)$ at $\alpha = 1$, construct the maximum envelope formed by the possible solution candidates for $\alpha \to 0$ under the condition of convexity.

This *general* part of our approach requires a total of maximum $2^n + (2n+1)s$ and minimum $2^n + 2ns$ function evaluations. It can be viewed as an analytical version of the *level interval algorithm* [6].

Example 1. The function $f_1 : \mathbb{R}^2 \to \mathbb{R}$ with

$$y_1 = f_1(x_1, x_2) = x_1^2 + x_2^2 - 5x_1 - x_2 \tag{9}$$

shall be evaluated for the two fuzzy numbers $\widetilde{x}_1 = \mathrm{tfn}(2, 2, 3)$ and $\widetilde{x}_2 = \mathrm{tfn}(2, 2, 2)$. Since

$$\begin{aligned} \frac{\partial f_1}{\partial x_1} &= 2x_1 - 5, \\ \frac{\partial f_1}{\partial x_2} &= 2x_2 - 1, \end{aligned} \tag{10}$$

the function f_1 is nonmonotonic in both x_1 and x_2 in the domain $\operatorname{supp}(\widetilde{x}_1) \times \operatorname{supp}(\widetilde{x}_2) = (0, 5) \times (0, 4)$ with one (global) extremum at $(x_{1,1}^*, x_{2,1}^*) = (2.5, 0.5) \in (0, 5) \times (0, 4)$. Hence, the general part of our approach should be applied. The solution candidates for $y_1(\alpha)$ are

$$\begin{aligned} y_1^{\mathrm{LL}}(\alpha) &= f_1\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{L}}(\alpha)\right) = 8\alpha^2 - 12\alpha, \\ y_1^{\mathrm{LR}}(\alpha) &= f_1\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{R}}(\alpha)\right) = 8\alpha^2 - 24\alpha + 12, \\ y_1^{\mathrm{RL}}(\alpha) &= f_1\left(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{L}}(\alpha)\right) = 13\alpha^2 - 17\alpha, \\ y_1^{\mathrm{RR}}(\alpha) &= f_1\left(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{R}}(\alpha)\right) = 13\alpha^2 - 29\alpha + 12, \\ y_1^{\mathrm{L1}}(\alpha) &= f_1\left(x_1^{\mathrm{L}}(\alpha), x_{2,1}^*\right) = 4\alpha^2 - 10\alpha - 0.25, \\ y_1^{\mathrm{R1}}(\alpha) &= f_1\left(x_1^{\mathrm{R}}(\alpha), x_{2,1}^*\right) = 9\alpha^2 - 15\alpha - 0.25, \\ y_1^{\mathrm{1L}}(\alpha) &= f_1\left(x_{1,1}^*, x_2^{\mathrm{L}}(\alpha)\right) = 4\alpha^2 - 2\alpha - 6.25, \\ y_1^{\mathrm{1R}}(\alpha) &= f_1\left(x_{1,1}^*, x_2^{\mathrm{R}}(\alpha)\right) = 4\alpha^2 - 14\alpha + 5.75, \\ y_1^{*1} &= f_1\left(x_{1,1}^*, x_{2,1}^*\right) = -6.5. \end{aligned} \tag{11}$$

We can see from their plots in Figure 1 that, starting from the modal point at $\alpha = 1$, the left branch of the maximum envelope, illustrated by the gray area, is formed by y_1^{RL} for $1 \geq \alpha > 0.8\overline{3}$, by y_1^{1L} for $0.8\overline{3} \geq \alpha > 0.25$, and by y_1^{*1} for $0.25 \geq \alpha > 0$, where the value $0.8\overline{3}$ corresponds to the intersection point between y_1^{RL} and y_1^{1L} and the value 0.25 to the intersection point between y_1^{1L} and y_1^{*1}. Its right branch, on the other hand, is entirely formed by y_1^{LR}. Hence, the α-cuts $y_1(\alpha) = [y_1^{\mathrm{L}}(\alpha), y_1^{\mathrm{R}}(\alpha)]$ of $\widetilde{y}_1$ are

$$\begin{aligned} y_1^{\mathrm{L}}(\alpha) &= \begin{cases} -6.5, & 0 < \alpha \leq 0.25, \\ 4\alpha^2 - 2\alpha - 6.25, & 0.25 < \alpha \leq 0.8\overline{3}, \\ 13\alpha^2 - 17\alpha, & 0.8\overline{3} < \alpha \leq 1, \end{cases} \\ y_1^{\mathrm{R}}(\alpha) &= 8\alpha^2 - 24\alpha + 12, \quad 0 < \alpha \leq 1. \end{aligned} \tag{12}$$

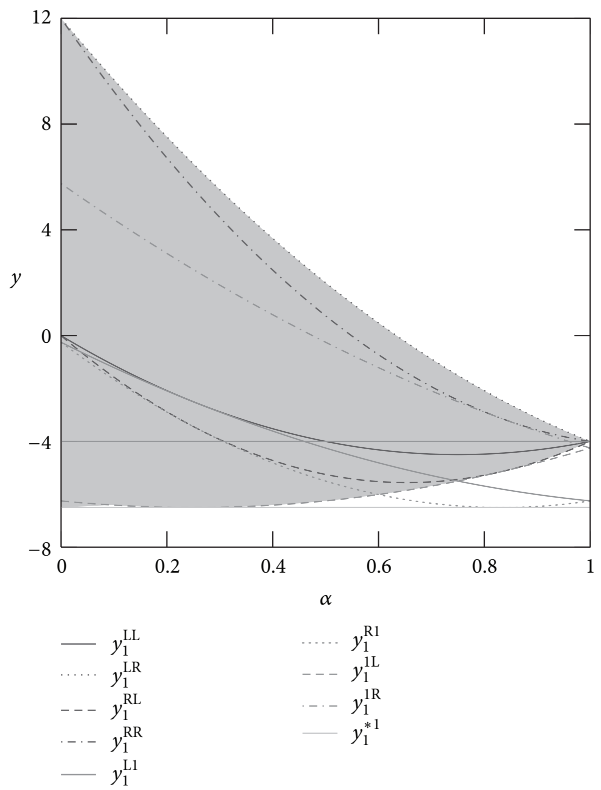

Figure 1: Solution candidates from Example 1.

With $y_1^{L}(0.25) = -6.5$, $y_1^{L}(0.8\overline{3}) = -5.13\overline{8}$, $y_1^{L}(1) = -4 = y_1^{R}(1)$, and $y_1^{R}(0) = 12$, the membership function of $\tilde{y}_1$ yields

$$\mu_{\tilde{y}_1}(y) = \begin{cases} \dfrac{1}{4} + \dfrac{1}{4}\sqrt{4y+26}, & -6.5 < y \le -5.13\overline{8}, \\ \dfrac{17}{26} + \dfrac{1}{26}\sqrt{52y+289}, & -5.13\overline{8} < y \le -4, \\ \dfrac{3}{2} - \dfrac{1}{4}\sqrt{2y+12}, & -4 < y < 12. \end{cases} \tag{13}$$

3.2. Extended Part. Let the continuous function f be (strictly) monotonic increasing in x_i, $i = 1,\ldots,k$, (strictly) monotonic decreasing in x_j, $j = 1,\ldots,\ell$, monotonic in x_p, $p = 1,\ldots,q$, and nonmonotonic in x_t, $t = 1,\ldots,u$, in the domain of interest, with $k+\ell+q+u = n$. Then, the analytical solution can be obtained as follows.

(1) Evaluate the function f for $x_i^{L}(\alpha)$, $i = 1,\ldots,k$, and $x_j^{R}(\alpha)$, $j = 1,\ldots,\ell$, including all the 2^{q+u} permutations of the interval boundaries of $x_p(\alpha) = [x_p^{L}(\alpha), x_p^{R}(\alpha)]$, $p = 1,\ldots,q$, and $x_t(\alpha) = [x_t^{L}(\alpha), x_t^{R}(\alpha)]$, $t = 1,\ldots,u$, to compute the monotonic solution candidates for $y^{L}(\alpha)$.

(2) Evaluate the function f for $x_i^{R}(\alpha)$, $i = 1,\ldots,k$, and $x_j^{L}(\alpha)$, $j = 1,\ldots,\ell$, including all the 2^{q+u} permutations of the interval boundaries of $x_p(\alpha) = [x_p^{L}(\alpha), x_p^{R}(\alpha)]$, $p = 1,\ldots,q$, and $x_t(\alpha) = [x_t^{L}(\alpha), x_t^{R}(\alpha)]$, $t = 1,\ldots,u$, to compute the monotonic solution candidates for $y^{R}(\alpha)$.

(3) Evaluate the function f for $x_i^{L}(\alpha)$, $i = 1,\ldots,k$, and $x_j^{R}(\alpha)$, $j = 1,\ldots,\ell$, including all the us combinations of the interval boundaries of $x_m(\alpha) = [x_m^{L}(\alpha), x_m^{R}(\alpha)]$, $m = k+\ell+q+1,\ldots,n$, with the components of the extreme points $(x_{1,r}^{*},\ldots,x_{u,r}^{*})$, $r = 1,\ldots,s$, to compute the nonmonotonic solution candidates for $y^{L}(\alpha)$.

(4) Evaluate the function f for $x_i^{R}(\alpha)$, $i = 1,\ldots,k$, and $x_j^{L}(\alpha)$, $j = 1,\ldots,\ell$, including all the us combinations of the interval boundaries of $x_m(\alpha) = [x_m^{L}(\alpha), x_m^{R}(\alpha)]$, $m = k+\ell+q+1,\ldots,n$ with the components of the extreme points $(x_{1,r}^{*},\ldots,x_{u,r}^{*})$, $r = 1,\ldots,s$, to compute the nonmonotonic solution candidates for $y^{R}(\alpha)$.

(5) Plot all solution candidates in the same diagram.

(6) Finally, starting from the modal point $\overline{y} = f(\overline{x}_1,\ldots,\overline{x}_n)$ at $\alpha = 1$, construct the maximum envelope formed by the possible solution candidates for $\alpha \to 0$ under the condition of convexity.

This *extended* part of our approach requires a total of $2^{q+u+1} + 2us$ function evaluations.

Example 2. Now, the function $f_2 : \mathbb{R}^2 \to \mathbb{R}$ with

$$y_2 = f_2(x_1, x_2) = x_1^2 + x_2^2 - 5x_1 \tag{14}$$

shall be evaluated for the two fuzzy numbers from Example 1. Since

$$\frac{\partial f_2}{\partial x_1} = 2x_1 - 5, \qquad \frac{\partial f_2}{\partial x_2} = 2x_2 > 0, \tag{15}$$

the function f_2 is nonmonotonic in x_1 with one (global) extremum at $x_{1,1}^{*} = 2.5$ and (strictly) monotonic increasing in x_1 in the domain $\operatorname{supp}(\tilde{x}_1) \times \operatorname{supp}(\tilde{x}_2) = (0,5) \times (0,4)$. Hence, the extended part of our approach should be applied. The monotonic solution candidates for $y_2^{L}(\alpha)$ are

$$y_2^{LL}(\alpha) = f_2\left(x_1^{L}(\alpha), x_2^{L}(\alpha)\right) = 8\alpha^2 - 10\alpha, \qquad y_2^{RL}(\alpha) = f_2\left(x_1^{R}(\alpha), x_2^{L}(\alpha)\right) = 13\alpha^2 - 15\alpha, \tag{16}$$

and for $y_2^{R}(\alpha)$,

$$y_2^{LR}(\alpha) = f_2\left(x_1^{L}(\alpha), x_2^{R}(\alpha)\right) = 8\alpha^2 - 26\alpha + 16, \qquad y_2^{RR}(\alpha) = f_2\left(x_1^{R}(\alpha), x_2^{R}(\alpha)\right) = 13\alpha^2 - 31\alpha + 16. \tag{17}$$

The nonmonotonic solution candidate for $y_2^{L}(\alpha)$ is

$$y_2^{1L}(\alpha) = f_2\left(x_{1,1}^{*}, x_2^{L}(\alpha)\right) = 4\alpha^2 - 6.25, \tag{18}$$

and for $y_2^{\mathrm{R}}(\alpha)$,

$$y_2^{1\mathrm{R}}(\alpha) = f_2\left(x_{1,1}^{*}, x_2^{\mathrm{R}}(\alpha)\right) = 4\alpha^2 - 16\alpha + 9.75. \quad (19)$$

We can see from their plots in Figure 2 that, starting from the modal point at $\alpha = 1$, the left branch of the maximum envelope is formed by y_2^{RL} for $1 \geq \alpha > 0.8\overline{3}$ and by $y_2^{1\mathrm{L}}$ for $0.8\overline{3} \geq \alpha > 0$, where the value $0.8\overline{3}$ corresponds to their intersection point. Its right branch, on the other hand, is entirely formed by y_2^{LR}. Hence, the α-cuts $y_2(\alpha) = [y_2^{\mathrm{L}}(\alpha), y_2^{\mathrm{R}}(\alpha)]$ of $\widetilde{y}_2$ are

$$y_2^{\mathrm{L}}(\alpha) = \begin{cases} 4\alpha^2 - 6.25, & 0 < \alpha \leq 0.8\overline{3}, \\ 13\alpha^2 - 15\alpha, & 0.8\overline{3} < \alpha \leq 1, \end{cases}$$
$$y_2^{\mathrm{R}}(\alpha) = 8\alpha^2 - 26\alpha + 16, \quad 0 < \alpha \leq 1. \quad (20)$$

With $y_2^{\mathrm{L}}(0) = -6.25$, $y_2^{\mathrm{L}}(0.8\overline{3}) = -3.47\overline{2}$, $y_2^{\mathrm{L}}(1) = -2 = y_2^{\mathrm{R}}(1)$, and $y_2^{\mathrm{R}}(0) = 16$, the membership function of $\widetilde{y}_2$ yields

$$\mu_{\widetilde{y}_2}(y) = \begin{cases} \dfrac{1}{4}\sqrt{4y + 25}, & -6.25 < y \leq -3.47\overline{2}, \\ \dfrac{15}{26} + \dfrac{1}{26}\sqrt{52y + 225}, & -3.47\overline{2} < y \leq -2, \\ \dfrac{13}{8} - \dfrac{1}{8}\sqrt{8y + 41}, & -2 < y < 16. \end{cases} \quad (21)$$

4. Engineering Application

In order to illustrate the analytical approach in a more practical context, we consider a linear system with one degree of freedom consisting of a block with mass m moving on a smooth surface as shown in Figure 3. The block is connected to a wall via a linear spring with spring constant k. This system is governed by the following linear, homogeneous ordinary differential equation of second order with constant coefficients [7]:

$$\ddot{x} + \omega^2 x = 0. \quad (22)$$

Here,

$$\omega = \sqrt{\frac{k}{m}} \quad (23)$$

denotes the *natural frequency* of the system. The general solution of (22) is given by

$$x(t) = x_0 \cos(\omega t) + \frac{\dot{x}_0}{\omega} \sin(\omega t), \quad (24)$$

where $x_0 = x(0)$ and $\dot{x}_0 = \dot{x}(0)$ denote the initial conditions.

We assume x_0 and ω to be uncertain, both described by fuzzy numbers. More specifically, the uncertain initial position is modeled by the (symmetric) triangular fuzzy number

$$\widetilde{x}_0 = \mathrm{tfn}(1, 0.5, 0.5)\ \mathrm{cm} \quad (25)$$

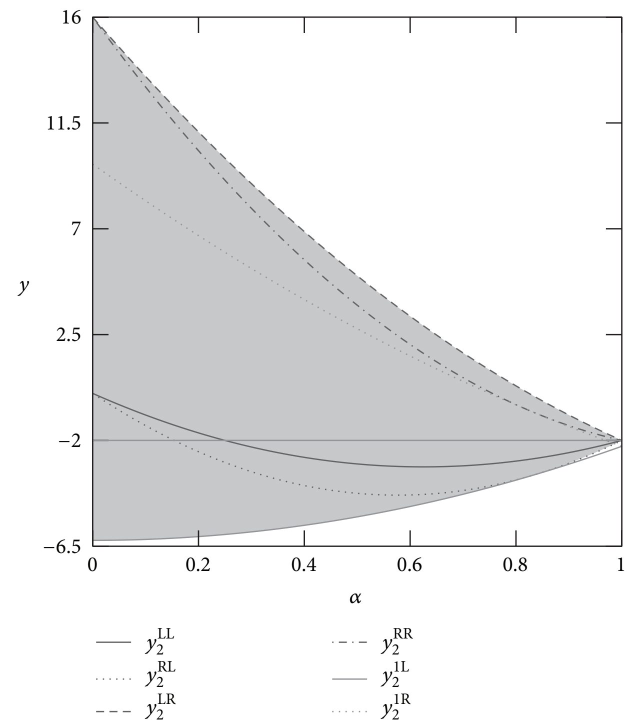

Figure 2: Solution candidates from Example 2.

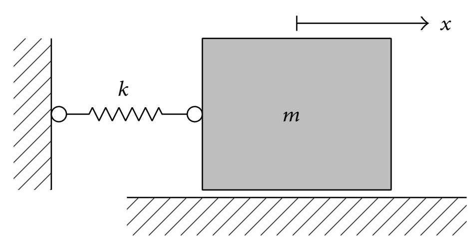

Figure 3: Mass-spring system.

and the uncertain natural frequency by the (symmetric) Gaussian fuzzy number

$$\widetilde{\omega} = \mathrm{gfn}(1, 0.05, 0.05)\ \mathrm{Hz}. \quad (26)$$

Furthermore, we assume $\dot{x}_0 = 0$. We are interested in the uncertain position of the mass after one period ($t = 2\pi$).

Since

$$\frac{\partial x}{\partial x_0} = \cos(2\pi\omega) \geq 0,$$
$$\frac{\partial x}{\partial \omega} = -2\pi x_0 \sin(2\pi\omega), \quad (27)$$

x is (strictly) monotonic increasing in x_0 and nonmonotonic in ω in the domain $\mathrm{supp}(\widetilde{x}_0) \times \mathrm{supp}(\widetilde{\omega}) \cap \mathbb{R}_+^2$ with an infinite number of local extrema $\omega_i^* = i/2$, $i \in \mathbb{N}_0$. Hence, the extended part of our approach should be applied. The monotonic solution candidates for $x^{\mathrm{L}}(\alpha)$ are

$$x^{\mathrm{LL}}(\alpha) = x^{\mathrm{LR}}(\alpha) = (1 - 0.5(1 - \alpha)) \cos\left(0.1\pi\sqrt{-2\ln(\alpha)}\right), \quad (28)$$

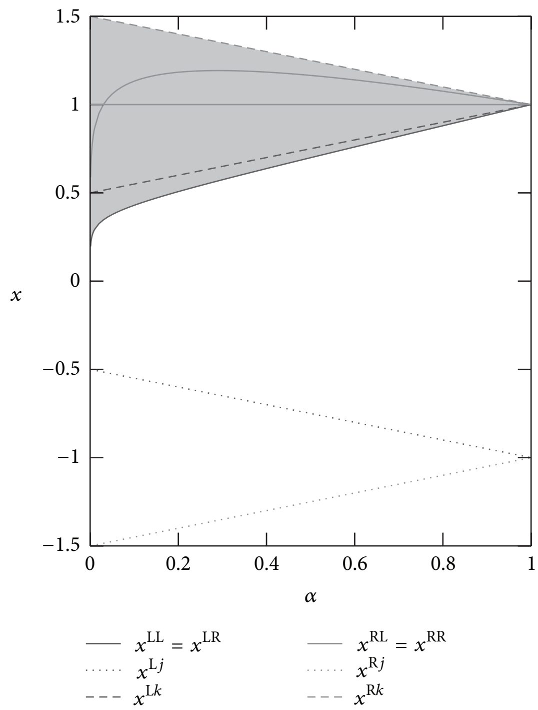

Figure 4: Solution candidates for $x(\alpha)$.

and for $x^{\mathrm{R}}(\alpha)$,

$$x^{\mathrm{RL}}(\alpha) = x^{\mathrm{RR}}(\alpha) = (1 + 0.5(1 - \alpha))\cos\left(0.1\pi\sqrt{-2\ln(\alpha)}\right). \tag{29}$$

The nonmonotonic solution candidates for $x^{\mathrm{L}}(\alpha)$, on the other hand, are

$$\begin{aligned} x^{\mathrm{L}j}(\alpha) &= -(1 - 0.5(1 - \alpha)), \quad j = 2\ell + 1,\ \ell \in \mathbb{N}_0, \\ x^{\mathrm{L}k}(\alpha) &= +(1 - 0.5(1 - \alpha)), \quad k = 2\ell + 2,\ \ell \in \mathbb{N}_0, \end{aligned} \tag{30}$$

and for $x^{\mathrm{R}}(\alpha)$,

$$\begin{aligned} x^{\mathrm{R}j}(\alpha) &= -(1 + 0.5(1 - \alpha)), \quad j = 2\ell + 1,\ \ell \in \mathbb{N}_0, \\ x^{\mathrm{R}k}(\alpha) &= +(1 + 0.5(1 - \alpha)), \quad k = 2\ell + 2,\ \ell \in \mathbb{N}_0. \end{aligned} \tag{31}$$

We can see from their plots in Figure 4 that the left branch of the maximum envelope is formed by $x^{\mathrm{LL}} = x^{\mathrm{LR}}$ and the right branch by $x^{\mathrm{R}k}$. Hence, the α-cuts $x(\alpha) = [x^{\mathrm{L}}(\alpha), x^{\mathrm{R}}(\alpha)]$ of $\tilde{x}$ are

$$\begin{aligned} x^{\mathrm{L}}(\alpha) &= (1 - 0.5(1 - \alpha))\cos\left(0.1\pi\sqrt{-2\ln(\alpha)}\right), \\ x^{\mathrm{R}}(\alpha) &= 1 + 0.5(1 - \alpha). \end{aligned} \tag{32}$$

Since $x^{\mathrm{L}}(\alpha)$ in (32) is not invertible with respect to α, it is not possible to give an analytical expression for the membership function of $\tilde{x}$. However, the α-cuts and the membership function are both equivalent representations of a fuzzy number. The inverted plots of (32) are illustrated in Figure 5.

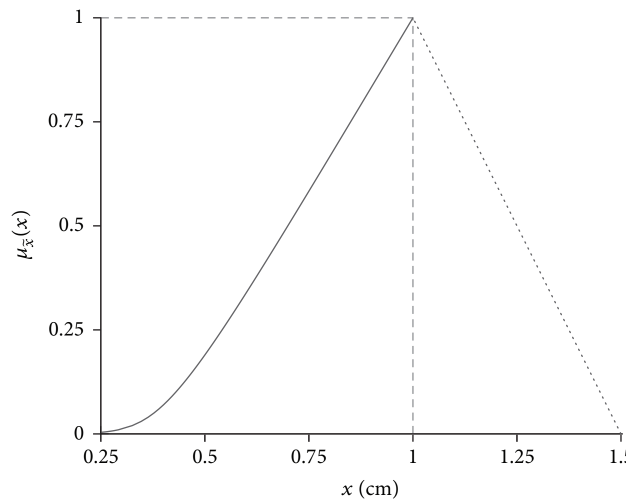

Figure 5: Membership function of $\tilde{x}$.

5. Conclusions

We extended our analytical approach from [1] to general, nonmonotonic functions of independent fuzzy numbers. It is based on an α-cut formulation of the extension principle and allows for the inclusion of parameter uncertainties into mathematical models.

In further research activities, the influence of interdependency may be a subject of investigation.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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