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Research Article

Common Coupled Fixed-Point Theorems in Generalized Fuzzy Metric Spaces

K. P. R. Rao,[1] I. Altun,[2] and S. Hima Bindu[3]

[1] *Department of Mathematics, Acharya Nagarjuna University, Dr. M.R. Appa Row Campus, Nuzvid 521 201, India*
[2] *Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Turkey*
[3] *Department of Mathematics, CH. S.D. St. Theresa's Junior College for Women, Eluru 534 001, India*

Correspondence should be addressed to K. P. R. Rao, kprrao2004@yahoo.com

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We prove two unique common coupled fixed-point theorems for self maps in symmetric G-fuzzy metric spaces.

1. Introduction and Preliminaries

Mustafa and Sims [1–3] and Naidu et al. [4] demonstrated that most of the claims concerning the fundamental topological structure of D-metric introduced by Dhage [5–8] and hence all theorems are incorrect. Alternatively, Mustafa and Sims [1, 2] introduced a G-metric space and obtained some fixed-point theorems in it. Some interesting references in G-metric spaces are [3, 9–15]. In this paper, we prove two unique common coupled fixed-point theorems for Jungck type and for three mappings in symmetric G-fuzzy metric spaces.

Before giving our main results, we recall some of the basic concepts and results in G-metric spaces and G-fuzzy metric spaces.

Definition 1 (see [2]). Let X be a nonempty set and let $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties:

(G_1) $G(x, y, z) = 0$ if $x = y = z$,

(G_2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G_3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, symmetry in all three variables,

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then, the function G is called a generalized metric or a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 2 (see [2]). The G-metric space (X, G) is called symmetric if $G(x, x, y) = G(x, y, y)$ for all $x, y \in X$.

Definition 3 (see [2]). Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence in X. A point $x \in X$ is said to be limit of $\{x_n\}$ if and only if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$. In this case, the sequence $\{x_n\}$ is said to be G-convergent to x.

Definition 4 (see [2]). Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence in X. $\{x_n\}$ is called G-Cauchy if and only if $\lim_{l,n,m\to\infty} G(x_l, x_n, x_m) = 0$. (X, G) is called G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 5 (see [2]). *In a G-metric space (X, G), the following are equivalent.*

(i) *The sequence $\{x_n\}$ is G-Cauchy.*

(ii) *For every $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq N$.*

Proposition 6 (see [2]). *Let (X, G) be a G-metric space. Then, the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Proposition 7 (see [2]). *Let (X, G) be a G-metric space. Then, for any $x, y, z, a \in X$, it follows that*

(i) *if $G(x, y, z) = 0$, then $x = y = z$,*

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(iii) $G(x, y, y) \le 2G(x, x, y)$,

(iv) $G(x, y, z) \le G(x, a, z) + G(a, y, z)$,

(v) $G(x, y, z) \le (2/3)[G(x, a, a) + G(y, a, a) + G(z, a, a)]$.

Proposition 8 (see [2]). *Let (X, G) be a G-metric space. Then, for a sequence $\{x_n\} \subseteq X$ and a point $x \in X$, the following are equivalent:*

(i) *$\{x_n\}$ is G-convergent to x,*

(ii) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$,

(iii) $G(x_n, x, x) \to 0$ *as* $n \to \infty$,

(iv) $G(x_m, x_n, x) \to 0$ *as* $m, n \to \infty$.

Recently, Sun and Yang [16] introduced the concept of G-fuzzy metric spaces and proved two common fixed-point theorems for four mappings.

Definition 9 (see [16]). A 3-tuple $(X, G, *)$ is called a G-fuzzy metric space if X is an arbitrary nonempty set, $*$ is a continuous t-norm, and G is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions for each $t, s > 0$:

(i) $G(x, x, y, t) > 0$ for all $x, y \in X$ with $x \neq y$,

(ii) $G(x, x, y, t) \ge G(x, y, z, t)$ for all $x, y, z \in X$ with $y \neq z$,

(iii) $G(x, y, z, t) = 1$ if and only if $x = y = z$,

(iv) $G(x, y, z, t) = G(p(x, y, z), t)$, where p is a permutation function,

(v) $G(x, y, z, t + s) \ge G(a, y, z, t) * G(x, a, a, s)$ for all $x, y, z, a \in X$,

(vi) $G(x, y, z, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Definition 10 (see [16]). A G-fuzzy metric space $(X, G, *)$ is said to be symmetric if $G(x, x, y, t) = G(x, y, y, t)$ for all $x, y \in X$ and for each $t > 0$.

Example 11. Let X be a nonempty set and let G be a G-metric on X. Denote $a * b = ab$ for all $a, b \in [0, 1]$. For each $t > 0$, $G(x, y, z, t) = t/(t + G(x, y, z))$ is a G-fuzzy metric on X.

Let $(X, G, *)$ be a G-fuzzy metric space. For $t > 0$, $0 < r < 1$, and $x \in X$, the set $B_G(x, r, t) = \{y \in X : G(x, y, y, t) > 1 - r\}$ is called an open ball with center x and radius r.

A subset A of X is called an open set if for each $x \in X$, there exist $t > 0$ and $0 < r < 1$ such that $B_G(x, r, t) \subseteq A$.

A sequence $\{x_n\}$ in G-fuzzy metric space X is said to be G-convergent to $x \in X$ if $G(x_n, x_n, x, t) \to 1$ as $n \to \infty$ for each $t > 0$. It is called a G-Cauchy sequence if $G(x_n, x_n, x_m, t) \to 1$ as $n, m \to \infty$ for each $t > 0$. X is called G-complete if every G-Cauchy sequence in X is G-convergent in X.

Lemma 12 (see [16]). *Let $(X, G, *)$ be a G-fuzzy metric space. Then, $G(x, y, z, t)$ is nondecreasing with respect to t for all $x, y, z \in X$.*

Lemma 13 (see [16]). *Let $(X, G, *)$ be a G-fuzzy metric space. Then, G is a continuous function on $X^3 \times (0, \infty)$.*

Now onwards, we assume the following condition:

$$\lim_{t \to \infty} G(x, y, z, t) = 1 \quad \forall x, y, z \in X. \tag{P}$$

Using (P), one can prove the following lemma.

Lemma 14. *Let $(X, G, *)$ be a G-fuzzy metric space. If there exists $k \in (0, 1)$ such that*

$$\min\{G(x, y, z, kt), G(u, v, w, kt)\} \ge \min\{G(x, y, z, t), G(u, v, w, t)\} \tag{1}$$

for all $x, y, z, u, v, w \in X$ and $t > 0$, then $x = y = z$ and $u = v = w$.

Definition 15 (see [17]). Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 16 (see [18]). Let X be a nonempty set. An element $(x, y) \in X \times X$ is called

(i) a coupled coincidence point of $F : X \times X \to X$ and $g : X \to X$ if $gx = F(x, y)$ and $gy = F(y, x)$,

(ii) a common coupled fixed point of $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Definition 17 (see [18]). Let X be a nonempty set. The mappings $F : X \times X \to X$ and $g : X \to X$ are called W-compatible if $g(F(x, y)) = F(gx, gy)$ and $g(F(y, x)) = F(gy, gx)$ whenever $gx = F(x, y)$ and $gy = F(y, x)$ for some $(x, y) \in X \times X$.

Now, we give our main results.

2. Main Results

Theorem 18. *Let $(X, G, *)$ be a G-fuzzy metric space with $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $S : X \times X \to X$ and let $f : X \to X$ be mappings satisfying*

$$G(S(x, y), S(u, v), S(u, v), kt) \ge \min\{G(fx, fu, fu, t), G(fy, fv, fv, t)\} \tag{2}$$

for all $x, y, u, v \in X$, where $0 \le k < 1$,

$$S(X \times X) \subseteq f(X) \text{ and } f(X) \text{ is a complete subspace of } X, \quad \text{the pair } (f, S) \text{ is } W\text{-compatible.} \tag{3}$$

Then S and f have a unique common coupled fixed point of the form (α, α) in $X \times X$.

Proof. Let $x_0, y_0 \in X$ and denote $z_n = S(x_n, y_n) = fx_{n+1}$, $p_n = S(y_n, x_n) = fy_{n+1}$, $n = 0, 1, 2, \ldots$. Let $d_n(t) =$

$G(z_n, z_{n+1}, z_{n+1}, t)$, $e_n(t) = G(p_n, p_{n+1}, p_{n+1}, t)$. From (2), we have

$$\begin{aligned} d_{n+1}(kt) &= G(z_{n+1}, z_{n+2}, z_{n+2}, kt) \\ &= G(S(x_{n+1}, y_{n+1}), S(x_{n+2}, y_{n+2}), S(x_{n+2}, y_{n+2}), kt) \\ &\geq \min\{G(z_n, z_{n+1}, z_{n+1}, t), G(p_n, p_{n+1}, p_{n+1}, t)\} \\ &\geq \min\{d_n(t), e_n(t)\}. \end{aligned} \tag{4}$$

Also,

$$\begin{aligned} e_{n+1}(kt) &= G(p_{n+1}, p_{n+2}, p_{n+2}, kt) \\ &= G(S(y_{n+1}, x_{n+1}), S(y_{n+2}, x_{n+2}), S(y_{n+2}, x_{n+2}), kt) \\ &\geq \min\{G(p_n, p_{n+1}, p_{n+1}, t), G(z_n, z_{n+1}, z_{n+1}, t)\} \\ &\geq \min\{e_n(t), d_n(t)\}. \end{aligned} \tag{5}$$

Thus, $\min\{d_{n+1}(kt), e_{n+1}(kt)\} \geq \min\{d_n(t), e_n(t)\}$. Hence,

$$\begin{aligned} &\min\{d_n(t), e_n(t)\} \\ &\quad \geq \min\left\{d_{n-1}\left(\frac{t}{k}\right), e_{n-1}\left(\frac{t}{k}\right)\right\} \\ &\quad \geq \min\left\{d_{n-2}\left(\frac{t}{k^2}\right), e_{n-2}\left(\frac{t}{k^2}\right)\right\} \\ &\quad \vdots \\ &\quad \geq \min\left\{d_0\left(\frac{t}{k^n}\right), e_0\left(\frac{t}{k^n}\right)\right\} \\ &\quad = \min\left\{G\left(z_0, z_1, z_1, \frac{t}{k^n}\right), G\left(p_0, p_1, p_1, \frac{t}{k^n}\right)\right\}. \end{aligned} \tag{6}$$

For any positive integer n and fixed positive integer p, we have

$$\begin{aligned} &G\left(z_n, z_{n+p}, z_{n+p}, t\right) \\ &\geq G\left(z_{n+p-1}, z_{n+p}, z_{n+p}, \frac{t}{p}\right) * G\left(z_{n+p-2}, z_{n+p-1}, z_{n+p-1}, \frac{t}{p}\right) \\ &\quad * \cdots * G\left(z_n, z_{n+1}, z_{n+1}, \frac{t}{p}\right) \\ &\geq \min\left\{G\left(z_0, z_1, z_1, \frac{t}{pk^{n+p-1}}\right), G\left(p_0, p_1, p_1, \frac{t}{pk^{n+p-1}}\right)\right\} \\ &\quad * \min\left\{G\left(z_0, z_1, z_1, \frac{t}{pk^{n+p-2}}\right), G\left(p_0, p_1, p_1, \frac{t}{pk^{n+p-2}}\right)\right\} \\ &\quad * \cdots * \min\left\{G\left(z_0, z_1, z_1, \frac{t}{pk^{n}}\right), G\left(p_0, p_1, p_1, \frac{t}{pk^{n}}\right)\right\}. \end{aligned} \tag{7}$$

Letting $n \to \infty$ and using (P), we get

$$\lim_{n\to\infty} G\left(z_n, z_{n+p}, z_{n+p}, t\right) \geq 1 * 1 * \cdots * 1 = 1. \tag{8}$$

Hence, $\lim_{n\to\infty} G(z_n, z_{n+p}, z_{n+p}, t) = 1$. Thus, $\{z_n\}$ is G-Cauchy in X. Similarly, we can show that $\{p_n\}$ is G-Cauchy in X. Since $f(X)$ is G-complete, $\{z_n\}$ and $\{p_n\}$ converge to some α and β in $f(X)$, respectively. Hence, there exist x and y in X such that $\alpha = fx$, $\beta = fy$:

$$\begin{aligned} &G(z_n, S(x, y), S(x, y), kt) \\ &\quad = G(S(x_n, y_n), S(x, y), S(x, y), kt) \\ &\quad \geq \min\{G(z_{n-1}, fx, fx, t), G(p_{n-1}, fy, fy, t)\}. \end{aligned} \tag{9}$$

Letting $n \to \infty$, we get

$$G(fx, S(x, y), S(x, y), kt) \geq \min\{1, 1\} = 1. \tag{10}$$

Hence, $S(x, y) = fx$. Similarly, it can be shown that $S(y, x) = fy$. Since (f, S) is W-compatible, we have

$$\begin{aligned} f\alpha &= ffx = f(S(x, y)) = S(fx, fy) = S(\alpha, \beta), \\ f\beta &= ffy = f(S(y, x)) = S(fy, fx) = S(\beta, \alpha). \end{aligned}$$

$$\begin{aligned} &G(z_n, f\alpha, f\alpha, kt) \\ &\quad = G(S(x_n, y_n), S(\alpha, \beta), S(\alpha, \beta), kt) \\ &\quad \geq \min\{G(z_{n-1}, f\alpha, f\alpha, t), G(p_{n-1}, f\beta, f\beta, t)\}. \end{aligned} \tag{11}$$

Letting $n \to \infty$, we get

$$G(\alpha, f\alpha, f\alpha, kt) \geq \min\{G(\alpha, f\alpha, f\alpha, t), G(\beta, f\beta, f\beta, t)\}. \tag{12}$$

Similarly, we can show that

$$G(\beta, f\beta, f\beta, kt) \geq \min\{G(\alpha, f\alpha, f\alpha, t), G(\beta, f\beta, f\beta, t)\}. \tag{13}$$

Thus,

$$\begin{aligned} &\min\{G(\alpha, f\alpha, f\alpha, kt), G(\beta, f\beta, f\beta, kt)\} \\ &\quad \geq \min\{G(\alpha, f\alpha, f\alpha, t), G(\beta, f\beta, f\beta, t)\}. \end{aligned} \tag{14}$$

From Lemma 14, we have $f\alpha = \alpha$ and $f\beta = \beta$. Thus, $\alpha = f\alpha = S(\alpha, \beta)$ and $\beta = f\beta = S(\beta, \alpha)$. Hence, (α, β) is a common coupled fixed point of S and f.

Suppose (α^1, β^1) is another common coupled fixed point of S and f:

$$\begin{aligned} G(\alpha, \alpha^1, \alpha^1, kt) &= G(S(\alpha, \beta), S(\alpha^1, \beta^1), S(\alpha^1, \beta^1), kt) \\ &\geq \min\{G(\alpha, \alpha^1, \alpha^1, t), G(\beta, \beta^1, \beta^1, t)\}. \end{aligned} \tag{15}$$

Similarly,

$$\begin{aligned} G(\beta, \beta^1, \beta^1, kt) &= G(S(\beta, \alpha), S(\beta^1, \alpha^1), S(\beta^1, \alpha^1), kt) \\ &\geq \min\{G(\alpha, \alpha^1, \alpha^1, t), G(\beta, \beta^1, \beta^1, t)\}. \end{aligned} \tag{16}$$

Thus,

$$\min\{G(\alpha,\alpha^1,\alpha^1,kt),G(\beta,\beta^1,\beta^1,kt)\} \geq \min\{G(\alpha,\alpha^1,\alpha^1,t),G(\beta,\beta^1,\beta^1,t)\}. \tag{17}$$

From Lemma 14, $\alpha^1 = \alpha$ and $\beta^1 = \beta$. Thus, (α,β) is the unique common coupled fixed point of S and f. Now, we will show that $\alpha = \beta$:

$$\begin{aligned} G(\alpha,\alpha,\beta,kt) &= G(S(\alpha,\beta),S(\alpha,\beta),S(\beta,\alpha),kt) \\ &\geq \min\{G(\alpha,\alpha,\beta,t),G(\beta,\beta,\alpha,t)\}, \\ G(\alpha,\beta,\beta,kt) &= G(S(\alpha,\beta),S(\beta,\alpha),S(\beta,\alpha),kt) \\ &\geq \min\{G(\alpha,\beta,\beta,t),G(\beta,\alpha,\alpha,t)\}. \end{aligned} \tag{18}$$

Thus,

$$\min\{G(\alpha,\alpha,\beta,kt),G(\alpha,\beta,\beta,kt)\} \geq \min\{G(\alpha,\alpha,\beta,t),G(\alpha,\beta,\beta,t)\}. \tag{19}$$

From Lemma 14, we have $\alpha = \beta$. Thus, α is a common fixed point of S and f, that is, $\alpha = f\alpha = S(\alpha,\alpha)$. Suppose α^1 is another common fixed point of S and f:

$$\begin{aligned} G(\alpha^1,\alpha,\alpha,t) &= G(S(\alpha^1,\alpha^1),S(\alpha,\alpha),S(\alpha,\alpha),t) \\ &\geq \min\left\{G\left(\alpha^1,\alpha,\alpha,\frac{t}{k}\right),G\left(\alpha^1,\alpha,\alpha,\frac{t}{k}\right)\right\} \\ &\geq G\left(\alpha^1,\alpha,\alpha,\frac{t}{k^2}\right) \\ &\vdots \\ &\geq G\left(\alpha^1,\alpha,\alpha,\frac{t}{k^n}\right) \longrightarrow 1 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{20}$$

Hence, $\alpha^1 = \alpha$. Thus, S and f have a unique common coupled fixed point of the form (α,α). □

Finally, we prove a common coupled fixed-point theorem for three mappings in symmetric G-fuzzy metric spaces.

Theorem 19. *Let $(X,G,*)$ be a symmetric G-complete fuzzy metric space with $a*b = \min\{a,b\}$ for all $a,b \in [0,1]$ and let $S,T,R : X\times X \to X$ be mappings satisfying*

$$\begin{aligned} &G(S(x,y),T(u,v),R(p,q),kt) \\ &\quad\geq \min\{G(x,u,p,t),G(y,v,q,t),G(x,x,S(x,y),t), \\ &\qquad G(u,u,T(u,v),t),G(p,p,R(p,q),t)\} \end{aligned} \tag{21}$$

for all $x,y,u,v,p,q \in X$, where $0 \leq k < 1$. Then, there exists $(x,y) \in X\times X$ such that

$$x = S(x,y) = T(x,y) = R(x,y), \tag{22}$$

$$y = S(y,x) = T(y,x) = R(y,x). \tag{23}$$

Or

$$\text{S, T, and R have a unique common coupled fixed point of the form (x,x) in $X\times X$.} \tag{24}$$

Proof. Let $x_0,y_0 \in X$. Define the sequences $\{x_n\}$ and $\{y_n\}$ in X as follows: $x_{3n+1} = S(x_{3n},y_{3n})$, $y_{3n+1} = S(y_{3n},x_{3n})$; $x_{3n+2} = T(x_{3n+1},y_{3n+1})$, $y_{3n+2} = T(y_{3n+1},x_{3n+1})$; $x_{3n+3} = R(x_{3n+2},y_{3n+2})$, $y_{3n+3} = R(y_{3n+2},x_{3n+2})$, $n = 0,1,2,\ldots$. Suppose $x_{3n+1} = x_{3n}$ for some n. Then, $S(x,y) = x$, where $x = x_{3n}$, $y = y_{3n}$. Suppose $T(x,y) \neq R(x,y)$. Then,

$$\begin{aligned} &G(x,T(x,y),R(x,y),kt) \\ &\quad= G(S(x,y),T(x,y),R(x,y),kt) \\ &\quad\geq \min\{1,1,1,G(x,x,T(x,y),t),G(x,x,R(x,y),t)\} \\ &\quad\geq G(x,T(x,y),R(x,y),t). \end{aligned} \tag{25}$$

It is a contradiction. Hence, $T(x,y) = R(x,y)$. From (25) and since X is symmetric,

$$\begin{aligned} G(x,T(x,y),T(x,y),kt) &\geq G(x,x,T(x,y),t) \\ &= G(x,T(x,y),T(x,y),t). \end{aligned} \tag{26}$$

From Lemma 14, we have $T(x,y) = x$. Thus, $S(x,y) = T(x,y) = R(x,y) = x$. Similarly, if $x_{3n+1} = x_{3n+2}$ or $x_{3n+2} = x_{3n+3}$, then also we can show that $S(x,y) = T(x,y) = R(x,y) = x$ for some x, y in X. Similarly, it can be shown that if $y_{3n} = y_{3n+1}$ or $y_{3n+1} = y_{3n+2}$ or $y_{3n+2} = y_{3n+3}$ then there exists $(x,y) \in X\times X$ such that

$$S(y,x) = T(y,x) = R(y,x) = y. \tag{27}$$

Now, assume that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$ for all n. Write $d_n(t) = G(x_n,x_{n+1},x_{n+2},t)$ and $e_n(t) = G(y_n,y_{n+1},y_{n+2},t)$:

$$\begin{aligned} &d_{3n}(kt) \\ &\quad= G(x_{3n},x_{3n+1},x_{3n+2},kt) \\ &\quad= G(S(x_{3n},y_{3n}),T(x_{3n+1},y_{3n+1}),R(x_{3n-1},y_{3n-1}),kt) \\ &\quad\geq \min\{d_{3n-1}(t),e_{3n-1}(t),G(x_{3n},x_{3n},x_{3n+1},t), \\ &\qquad G(x_{3n+1},x_{3n+1},x_{3n+2},t),G(x_{3n-1},x_{3n-1},x_{3n},t)\} \\ &\quad\geq \min\{d_{3n-1}(t),e_{3n-1}(t),d_{3n}(t),d_{3n}(t),d_{3n-1}(t)\}. \end{aligned} \tag{28}$$

Thus, $d_{3n}(kt) \geq \min\{d_{3n-1}(t),e_{3n-1}(t)\}$. Similarly, we have $e_{3n}(kt) \geq \min d_{3n-1}(t),e_{3n-1}(t)$.

Thus,

$$\min\{d_{3n}(kt),e_{3n}(kt)\} \geq \min\{d_{3n-1}(t),e_{3n-1}(t)\}. \tag{29}$$

Similarly, we can show that

$$\begin{aligned} \min\{d_{3n+1}(kt),e_{3n+1}(kt)\} &\geq \min\{d_{3n}(t),e_{3n}(t)\}, \\ \min\{d_{3n+2}(kt),e_{3n+2}(kt)\} &\geq \min\{d_{3n+1}(t),e_{3n+1}(t)\}. \end{aligned} \tag{30}$$

Thus,

$$\min\{d_{n+1}(kt), e_{n+1}(kt)\} \geq \min\{d_n(t), e_n(t)\}. \quad (31)$$

Hence

$$\begin{aligned} &\min\{d_n(t), e_n(t)\} \\ &\quad \geq \min\left\{d_n\left(\frac{t}{k}\right), e_n\left(\frac{t}{k}\right)\right\} \\ &\quad \geq \min\left\{d_n\left(\frac{t}{k^2}\right), e_n\left(\frac{t}{k^2}\right)\right\} \\ &\quad \vdots \\ &\quad \geq \min\left\{d_0\left(\frac{t}{k^n}\right), e_0\left(\frac{t}{k^n}\right)\right\} \\ &\quad = \min\left\{G\left(x_0, x_1, x_2, \frac{t}{k^n}\right), G\left(y_0, y_1, y_2, \frac{t}{k^n}\right)\right\}. \end{aligned} \quad (32)$$

Thus,

$$\begin{aligned} &G(x_n, x_{n+1}, x_{n+2}, t) \\ &\quad \geq \min\left\{G\left(x_0, x_1, x_2, \frac{t}{k^n}\right), G\left(y_0, y_1, y_2, \frac{t}{k^n}\right)\right\}. \end{aligned} \quad (33)$$

From (G_3), we have

$$\begin{aligned} &G(x_n, x_n, x_{n+1}, t) \\ &\quad \geq G(x_n, x_{n+1}, x_{n+2}, t) \\ &\quad \geq \min\left\{G\left(x_0, x_1, x_2, \frac{t}{k^n}\right), G\left(y_0, y_1, y_2, \frac{t}{k^n}\right)\right\}. \end{aligned} \quad (34)$$

As in Theorem 18, we can show that $\{x_n\}$ and $\{y_n\}$ are G-Cauchy sequences in X. Since X is G-complete, there exist $x, y \in X$ such that $x_n \to x$ and $y_n \to y$:

$$\begin{aligned} &G(S(x,y), x_{3n+2}, x_{3n+3}, kt) \\ &\quad = G(S(x,y), T(x_{3n+1}, y_{3n+1}), R(x_{3n+2}, y_{3n+2}), kt) \\ &\quad \geq \min\{G(x, x_{3n+1}, x_{3n+2}, t), G(y, y_{3n+1}, y_{3n+2}, t), \\ &\qquad G(x, x, S(x,y), t), G(x_{3n+1}, x_{3n+1}, x_{3n+2}, t), \\ &\qquad G(x_{3n+2}, x_{3n+2}, x_{3n+3}, t)\}. \end{aligned} \quad (35)$$

Letting $n \to \infty$,

$$\begin{aligned} G(S(x,y), x, x, kt) &\geq \min\{1, 1, G(x, x, S(x,y), t), 1, 1\} \\ &= G(x, x, S(x,y), t). \end{aligned} \quad (36)$$

From this, we have $S(x,y) = x$. As in the first part of proof, we can show that $S(x,y) = T(x,y) = R(x,y) = x$. Similarly, it can be shown that $S(y,x) = T(y,x) = R(y,x) = y$. Thus, (x,y) is a common coupled fixed point of S, T, and R. Suppose (x^1, y^1) is another common coupled fixed point of S, T, and R. Consider

$$\begin{aligned} G(x, x, x^1, kt) &= G(S(x,y), T(x,y), R(x^1, y^1), kt) \\ &\geq \min\{G(x, x, x^1, t), G(y, y, y^1, t), 1, 1, 1\} \\ &= \min\{G(x, x, x^1, t), G(y, y, y^1, t)\}. \end{aligned} \quad (37)$$

Also,

$$\begin{aligned} G(y, y, y^1, kt) &= G(S(y,x), T(y,x), R(y^1, x^1), kt) \\ &\geq \min\{G(x, x, x^1, t), G(y, y, y^1, t), 1, 1, 1\} \\ &= \min\{G(x, x, x^1, t), G(y, y, y^1, t)\}. \end{aligned} \quad (38)$$

Thus,

$$\begin{aligned} &\min\{G(x, x, x^1, kt), G(y, y, y^1, kt)\} \\ &\quad \geq \min\{G(x, x, x^1, t), G(y, y, y^1, t)\}. \end{aligned} \quad (39)$$

From Lemma 14, we have $x^1 = x$ and $y^1 = y$. Thus, (x,y) is the unique common coupled fixed point of S, T, and R. Now, we will show that $x = y$. Consider

$$\begin{aligned} G(x, x, y, kt) &= G(S(x,y), T(x,y), R(y,x), kt) \\ &\geq \min\{G(x, x, y, t) G(y, y, x, t), 1, 1, 1\} \\ &= G(x, x, y, t). \end{aligned} \quad (40)$$

Hence, $x = y$. Thus, S, T, and R have a unique common coupled fixed point of the form (x,x). □

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