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Research Article

On Categories of Fuzzy Petri Nets

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We introduce the concepts of fuzzy Petri nets and marked fuzzy Petri nets along with their appropriate morphisms, which leads to two categories of such Petri nets. Some aspects of the internal structures of these categories are then explored, for example, their reflectiveness/coreflectiveness and symmetrical monoidal closed structure.

1. Introduction

Petri nets are a well-known model of concurrent systems [1]. A number of authors have been led to the study of the various *categories* of Petri nets and their appropriate morphisms [2–4], in the belief that the categorical study provides a tool to compare different models of Petri nets. At the same time, *fuzzy* Petri nets have also been studied by many authors in different ways [5, 6]. In this paper, taking a cue from [3], we introduce another concept of fuzzy Petri nets (although we model our definition on the lines of [2]). This, along with their appropriate morphisms, results in a category of fuzzy Petri nets (and also of *marked* fuzzy Petri nets). The structure of these categories is then studied on the lines similar to those in [2], showing that one of the categories is *symmetric monoidal closed*.

2. Category of Fuzzy Petri Nets

For categorical concepts used here, [7] may be referred. We begin by collecting some basic definitions.

A Petri net is a bipartite graph, consisting of two kinds of nodes, namely, places and transitions, where arcs are either from a place to a transition or vice versa [3]. Graphically, the places are represented by circles, transitions by rectangles, and the arcs by arrows. We define a fuzzy Petri net as follows.

Definition 1. A *fuzzy Petri net* (in short, fPn) is 4-tuple $N = (P, T, I^-, I^+)$, where P and T are sets, called the set of places and set of transitions, respectively, and $I^-, I^+ : P \times T \to [0, 1]$, are functions, called the incidence functions.

We may interpret I^- and I^+ defined above as follows.

For $(p, t) \in P \times T$, $I^-(p, t)$ (resp., $I^+(p, t)$) gives, the grade with which place p is related to transition t (resp., transition t is related to place p). Thus, I^- and I^+ describe some kind of *fuzzy arcs* between places and transitions.

Definition 2. An fPn-*morphism* from an fPn (P_1, T_1, I_1^-, I_1^+) to fPn (P_2, T_2, I_2^-, I_2^+) is a pair (f, g) of functions, $f : P_2 \to P_1$ and $g : T_1 \to T_2$ such that the following two diagrams:

$$\begin{array}{ccc} P_2 \times T_1 & \xrightarrow{(f,1)} & P_1 \times T_1 \\ {\scriptstyle (1,g)}\downarrow & \leq & \downarrow{\scriptstyle I_1^-} \\ P_2 \times T_2 & \xrightarrow{I_2^-} & [0,1] \end{array} \qquad \begin{array}{ccc} P_2 \times T_1 & \xrightarrow{(f,1)} & P_1 \times T_1 \\ {\scriptstyle (1,g)}\downarrow & \geq & \downarrow{\scriptstyle I_1^+} \\ P_2 \times T_2 & \xrightarrow{I_2^+} & [0,1] \end{array} \quad (1)$$

"hold", by which it is meant that for all $(p, t) \in P_2 \times T_1$, $I_1^-(f(p), t) \geq I_2^-(p, g(t))$ and $I_1^+(f(p), t) \leq I_2^+(p, g(t))$.

Remark 3. Fuzzy Petri nets and fPn-morphisms form a category, denoted as FPN (the identity morphisms and the composition for this category are obvious to guess).

Definition 4. The *product* of two fPn's $N_1 = (P_1, T_1, I_1^-, I_1^+)$ and $N_2 = (P_2, T_2, I_2^-, I_2^+)$ is the fPn $N_1 \times N_2 = (\widetilde{P}_1 \cup \widetilde{P}_2, T_1 \times T_2, I^-, I^+)$, where $\widetilde{P}_1 = P_1 \times \{0\}, \widetilde{P}_2 = P_2 \times \{1\}$ and $I^-, I^+ : (\widetilde{P}_1 \cup \widetilde{P}_2) \times (T_1 \times T_2) \to [0,1]$ are given by

$$
\begin{aligned}
I^-((p,n),(t_1,t_2)) &= \begin{cases} I_1^-(p,t_1) & \text{if } n = 0, \\ I_2^-(p,t_2) & \text{if } n = 1, \end{cases} \\
I^+((p,n),(t_1,t_2)) &= \begin{cases} I_1^+(p,t_1) & \text{if } n = 0, \\ I_2^+(p,t_2) & \text{if } n = 1, \end{cases}
\end{aligned} \tag{2}
$$

for all $(p,n) \in \widetilde{P}_1 \cup \widetilde{P}_2$ and $(t_1,t_2) \in T_1 \times T_2$.

Proposition 5. *Let $\pi_i : T_1 \times T_2 \to T_i, i = 1,2$, be the two projections and let $\rho_i : P_i \to \widetilde{P}_1 \cup \widetilde{P}_2, i = 1,2$, be the two injections. Then $(\rho_i, \pi_i) : N_1 \times N_2 \to N_i, i = 1,2$, are fPn-morphisms.*

Proof. To prove that $(\rho_1, \pi_1) : N_1 \times N_2 \to N_1$ is an fPn-morphism, we need to show that the following two diagrams hold.

$$
\begin{array}{ccc}
\tilde{P}_1 \cup \tilde{P}_2 \times T' & \overset{(f,1)}{\longrightarrow} & P' \times T' \\
{\scriptstyle (1,g)}\big\downarrow & \leq & {\scriptstyle I'^-}\big\downarrow \\
(\tilde{P}_1 \cup \tilde{P}_2) \times (T_1 \times T_2) & \overset{I^-}{\longrightarrow} & [0,1] \\
\tilde{P}_1 \cup \tilde{P}_2 \times T' & \overset{(f,1)}{\longrightarrow} & P' \times T' \\
{\scriptstyle (1,g)}\big\downarrow & \geq & {\scriptstyle I'^+}\big\downarrow \\
(\tilde{P}_1 \cup \tilde{P}_2) \times (T_1 \times T_2) & \overset{I^+}{\longrightarrow} & [0,1].
\end{array} \tag{3}
$$

The above diagrams hold because for all $(p_1,(t_1,t_2)) \in P_1 \times (T_1 \times T_2), I^-(\rho_1(p_1),(t_1,t_2)) = I^-((p_1,0),(t_1,t_2)) = I_1^-(p_1,t_1) = I_1^-(p_1,\pi_1(t_1,t_2))$ and $I^+(\rho_1(p_1),(t_1,t_2)) = I^+((p_1,0),(t_1,t_2)) = I_1^+(p_1,t_1) = I_1^+(p_1,\pi_1(t_1,t_2))$. Similarly, one can prove that $(\rho_2,\pi_2) : N_1 \times N_2 \to N_2$ is also an fPn-morphism. □

Proposition 6. *The product $N_1 \times N_2$ of fPn's N_1 and N_2 is the categorical product of N_1 and N_2 in FPN.*

Proof. Let $N' = (P', T', I'^-, I'^+)$ be an fPn together with fPn-morphisms $(f_i, g_i) : N' \to N_i, i = 1,2$. We show that there exists a unique fPn-morphism $(f,g) : N' \to N_1 \times N_2$ such that $(\rho_i,\pi_i) \circ (f,g) = (f_i,g_i), i = 1,2$, or equivalently that $f \circ \rho_i = f_i$ and $\pi_i \circ g = g_i, i = 1,2$. For this purpose, we choose the following f and g. Let $f : \widetilde{P}_1 \cup \widetilde{P}_2 \to P'$ be the map given by

$$
f((p,n)) = \begin{cases} f_1(p), & \text{if } n = 0, \\ f_2(p), & \text{if } n = 1, \end{cases} \tag{4}
$$

and $g = (g_1, g_2)$. We show that the diagrams

$$
\begin{array}{ccc}
P_1 \times (T_1 \times T_2) & \overset{(\rho_1,1)}{\longrightarrow} & (\tilde{P}_1 \cup \tilde{P}_2) \times (T_1 \times T_2) \\
{\scriptstyle (1,\pi_1)}\big\downarrow & \leq & {\scriptstyle I^-}\big\downarrow \\
P_1 \times T_1 & \overset{I_2^-}{\longrightarrow} & [0,1] \\
P_1 \times (T_1 \times T_2) & \overset{(\rho_1,1)}{\longrightarrow} & (\tilde{P}_1 \cup \tilde{P}_2) \times (T_1 \times T_2) \\
{\scriptstyle (1,\pi_1)}\big\downarrow & \geq & {\scriptstyle I^+}\big\downarrow \\
P_1 \times T_1 & \overset{I_2^+}{\longrightarrow} & [0,1]
\end{array} \tag{5}
$$

hold, that is, for all $((p,n),t) \in (\widetilde{P}_1 \cup \widetilde{P}_2) \times T'$, $I'^-(f((p,n)),t) \geq I^-(p,g(t))$ and $I'^+(f((p,n)),t) \leq I^+(p,g(t))$, that is, $I_1^-(p,g_1(t)) \leq I'^-(f_1(p),t)$, if $n = 0$ and $I_2^-(p,g_2(t)) \leq I'^-(f_2(p),t)$, if $n = 1$, for all $t \in T'$ as well as $I_1^+(p,g_1(t)) \geq I'^+(f_1(p),t)$, if $n = 0$ and $I_2^+(p,g_2(t)) \geq I'^+(f_2(p),t)$, if $n = 1$, for all $t \in T'$. But as $(f_i,g_i) : N' \to N_i, i = 1,2$, are fPn-morphisms, the above inequalities hold, whereby the diagrams (5) hold. Thus, $(f,g) : N' \to N_1 \times N_2$ is an fPn-morphism. Also, the definitions of f and g are such that we obviously have $f \circ \rho_i = f_i$ and $\pi_i \circ g = g_i, i = 1,2$.

To prove the uniqueness of (f,g), let there exist another fPn-morphism (f',g') such that $(\rho_i,\pi_i) \circ (f',g') = (f_i,g_i), i = 1,2$, that is, $f' \circ \rho_i = f_i$ and $\pi_i \circ g' = g_i, i = 1,2$. We then have $f' \circ \rho_1 = f \circ \rho_1, f' \circ \rho_2 = f \circ \rho_2, \pi_1 \circ g' = \pi_1 \circ g$, and $\pi_2 \circ g' = \pi_2 \circ g$, whereby $f = f'$ and $g = g'$. Thus, $(f',g') = (f,g)$, proving the uniqueness of (f,g). Hence the product is a categorical product. □

Definition 7. The *coproduct* of two fPn's $N_1 = (P_1, T_1, I_1^-, I_1^+)$ and $N_2 = (P_2, T_2, I_2^-, I_2^+)$ is the fPn $N_1 \oplus N_2 = (P_1 \times P_2, \widetilde{T}_1 \cup \widetilde{T}_2, I^-, I^+)$, where $\widetilde{T}_1 = T_1 \times \{0\}, \widetilde{T}_2 = T_2 \times \{1\}$, and $I^-, I^+ : (P_1 \times P_2) \times (\widetilde{T}_1 \cup \widetilde{T}_2) \to [0,1]$ are given by

$$
\begin{aligned}
I^-((p_1,p_2),(t,n)) &= \begin{cases} I_1^-(p_1,t), & \text{if } n = 0, \\ I_2^-(p_2,t), & \text{if } n = 1, \end{cases} \\
I^+((p_1,p_2),(t,n)) &= \begin{cases} I_1^+(p_1,t), & \text{if } n = 0, \\ I_2^+(p_2,t), & \text{if } n = 1, \end{cases}
\end{aligned} \tag{6}
$$

for all $(p_1,p_2) \in P_1 \times P_2$ and $(t,n) \in \widetilde{T}_1 \cup \widetilde{T}_2$.

Similar to Propositions 5 and 6, the following two propositions can also be proved.

Proposition 8. *Let $\pi_i : P_1 \times P_2 \to P_i, i = 1,2$, be the two projections and $\rho_i : T_i \to \widetilde{T}_1 \cup \widetilde{T}_2, i = 1,2$, be the two injections. Then $(\pi_i,\rho_i) : N_i \to N_1 \oplus N_2, i = 1,2$, are fPn-morphisms.*

Proposition 9. *The coproduct $N_1 \oplus N_2$ of fPn's N_1 and N_2 is the categorical coproduct of N_1 and N_2 in FPN.*

Definition 10. Given two fPn's $N_1 = (P_1, T_1, I_1^-, I_1^+)$ and $N_2 = (P_2, T_2, I_2^-, I_2^+)$, we define two new fPn's $N_1 \otimes N_2$ and $N_1^{N_2}$

as follows (for sets X and Y, Y^X shall denote the set of all functions from X to Y):

(1) $N_1 \otimes N_2 = (P_1^{T_2} \times P_2^{T_1}, T_1 \times T_2, I^-, I^+)$, where I^-, $I^+ : (P_1^{T_2} \times P_2^{T_1}) \times (T_1 \times T_2) \to [0,1]$ are defined as

$$I^-((\alpha,\beta),(t_1,t_2)) = \wedge\{1, I_1^-(\alpha(t_2),t_1) + I_2^-(\beta(t_1),t_2)\}$$
$$I^+((\alpha,\beta),(t_1,t_2)) = \vee\{0, I_2^+(\beta(t_1),t_2) - I_1^+(\alpha(t_2),t_1)\},$$
$$\forall((\alpha,\beta),(t_1,t_2)) \in \left(P_1^{T_2} \times P_2^{T_1}\right) \times (T_1 \times T_2). \tag{7}$$

(2) $N_1^{N_2} = (T_2 \times P_1, T_1^{T_2} \times P_2^{P_1}, I^-, I^+)$, where $I^-, I^+ : (T_2 \times P_1) \times (T_1^{T_2} \times P_2^{P_1}) \to [0,1]$ are defined as

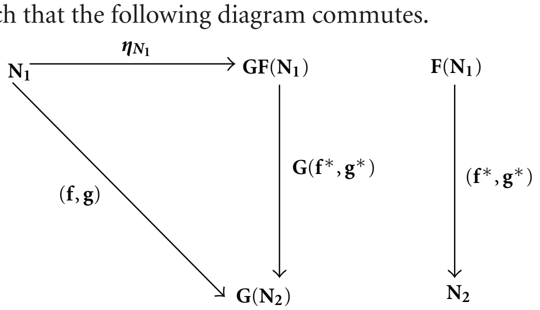
$$I^-((t,p),(\alpha,\beta)) = \vee\{0, I_1^-(p,\alpha(t)) - I_2^-(\beta(p),t)\}$$
$$I^+((t,p),(\alpha,\beta)) = \wedge\{1, I_1^+(p,\alpha(t)) + I_2^+(\beta(p),t)\} \tag{8}$$
$$\forall((t,p),(\alpha,\beta)) \in (T_2 \times P_1) \times \left(T_1^{T_2} \times P_2^{P_1}\right).$$

Proposition 11. *The category FPN is a symmetric monoidal closed category (with the constructions in (1) and (2) above, respectively, giving the associated tensor product and hom-object).*

Proof. For convenience, the notation N_i is used to denote the fPn (P_i, T_i, I_i^-, I_i^+). We give a sketch of the proof of closedness of FPN. For this, the two functors $- \otimes N_0, (-)^{N_0} : FPN \to FPN$ are denoted, respectively, as F and G, which map any FPN-morphism $(f,g) : N_1 \to N_2$, respectively, to FPN-morphisms, $(f',g') : F(N_1) \to F(N_2)$, and $(f''g'') : G(N_1) \to G(N_2)$, such that $f'((\lambda,\mu)) = (f \circ \lambda, \mu \circ g)$, $g'((t_2,t_0)) = (g(t_2),t_0)$, for all $((\lambda,\mu),(t_2,t_0)) \in (P_2^{T_0} \times P_0^{T_2}) \times (T_2 \times T_0)$, and $f''((t_0,p_2)) = (t_0, f(p_2)), g''((\rho,\nu)) = (g \circ \rho, \nu \circ f)$, for all $((t_0,p_2),(\rho,\nu)) \in (T_0 \times P_2) \times (T_1^{T_0} \times P_0^{P_1})$.

It turns out that G is a right adjoint to F; the associated unit of the adjunction, $\eta : I_{FPN} \to GF$, is given for each $N_1 \in FPN$, by $\eta_{N_1} = (F_{N_1}, G_{N_1}) : N_1 \to GF(N_1)$, where $F_{N_1} : T_0 \times (P_1^{T_0} \times P_0^{T_1}) \to P_1$ and $G_{N_1} : T_1 \to (T_1 \times T_0)^{T_0} \times P_0^{(P_1^{T_0} \times P_0^{T_1})}$, are such that $F_{N_1}((t_0,(\lambda,\mu)) = \lambda(t_0)$, and $G_{N_1}(t_1) = (\alpha^{t_1}, \beta^{t_1})$, with $\alpha^{t_1}(t_0) = (t_1,t_0)$ and $\beta^{t_1}((\lambda,\mu)) = \mu(t_1)$, for all $(t_0,(\lambda,\mu)) \in T_0 \times (P_1^{T_0} \times P_0^{T_1})$, and for all $t_1 \in T_1$.

To establish the universality of η_{N_1}, we need to produce, for any given $N_2 \in$ FPN and FPN-morphism $(f,g) : N_1 \to G(N_2)$, a unique FPN-morphism $(f^*,g^*) : F(N_1) \to N_2$, such that the following diagram commutes.



We only describe (f^*,g^*), leaving out the verification of the commutativity of the above diagram and the uniqueness of (f^*,g^*). (f^*,g^*) is given by $f^* : P_2 \to P_1^{T_0} \times P_0^{T_1}$ and $g^* : T_1 \times T_0 \to T_2$, such that $f^*(p_2) = (f_1^{p_2}, f_2^{p_2})$ and $g^*(t_1,t_0) = g_1^{t_1}(t_0)$, with $f_1^{p_2}(t_0) = f(t_0), f_2^{p_2}(t_1) = g_2^{t_1}(p_2)$, and $g(t_1) = (g_1^{t_1}, g_2^{t_1})$. □

3. Marked Fuzzy Petri Nets

A marked Petri net is a Petri net together with a function, called *marking* defined from the set of places to the set of natural numbers [2]. Marking at a particular place gives the number of *tokens* at that particular place. In this section, we introduce a concept of marked fuzzy Petri nets and thereby a category of marked fuzzy Petri nets.

Definition 12. An fPn $N = (P,T,I^-,I^+)$, together with a function $M : P \to [0,1]$ (called a *fuzzy marking* of N), is called a *marked fuzzy Petri net* (in short, an mfPn) and is denoted as (N,M).

Here, marking at a particular place may be interpreted as the degree of confidence to which a token can reside at that place.

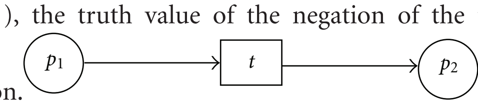
Definition 13. Given an mfPn (N,M), a transition $t \in T$ is said to *fire at* M (or t is *enabled* at M), if $I^-(p,t) \le M(p)$, for all $p \in P$.

In an mfPn $(P,T,I^-,I^+,M), I^- : P \times T \to [0,1]$, for fixed $t \in T$ induces a function $I_t^- : P \to [0,1]$ such that for $p \in P, I_t^-(p)$, gives the degree of confidence to which a transition $t \in T$ can fire at marking M. Thus, a transition t at a marking M of mfPn (N,M) can fire if the degree of confidence to which it fires does not exceed the degree of confidence to which a token can reside at places.

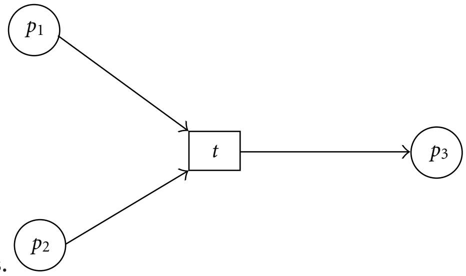
After t firing at the fuzzy marking M, we get a new fuzzy marking M_t of N, given by $M_t(p) = \min\{1, M(p) - I^-(p,t) + I^+(p,t)\}$, for all $p \in P$. We say that $t \in T$ fires at M to *yield* M_t and denote this by $M \xrightarrow{t} M_t$. Also, M_t is then said to be *directly reachable* from M through the transition t.

Similar to [8], the marked fuzzy Petri net models of negation, disjunction, and conjunction of fuzzy proposition, can also be given. We illustrate these by following examples.

Example 14. Consider the following graphical representation of an mfPn, which gives the truth value of the negation of a fuzzy proposition. For this, take an mfPn with $P = \{p_1, p_2\}$ and $T = \{t\}$. Given a fuzzy proposition, the initial marking M is so chosen that $M(p_1)$ is the truth value of the fuzzy proposition and $M(p_2) = 0$. Also, $I^-(p_1,t)$ is so chosen that $I^-(p_1,t) \le M(p_1)$ (so that the transition t can fire) and we also take $I^+(p_2,t)$ to be $1 - M(p_1)$. After the firing of the transition t, at marking M, the marking at p_2 is given by $M_t(p_2) = \min\{1, M(p_2) - I^-(p_2,t) + I^+(p_2,t)\} = 1 - M(p_1)$, the truth value of the negation of the fuzzy proposition.



Example 15. Similar to Example 14, consider the following graphical representation of an mfPn, which gives the disjunction of truth values of two fuzzy propositions. For this, take an mfPn, with $P = \{p_1, p_2, p_3\}$ and $T = \{t\}$. Given two fuzzy propositions, the initial marking M is so chosen that $M(p_1)$ and $M(p_2)$ are the truth values of the given fuzzy propositions and $M(p_3) = 0$. Also, $I^-(p_1, t)$ and $I^-(p_2, t)$ are so chosen that $I^-(p_1, t) \le M(p_1)$ and $I^-(p_2, t) \le M(p_2)$ (so that the transition t can fire) and we also take $I^+(p_3, t)$ to be $M(p_1) \vee M(p_2)$. After the firing of the transition t, at marking M, the marking at p_3 is given by $M_t(p_3) = \min\{1, M(p_3) - I^-(p_3, t) + I^+(p_3, t)\} = M(p_1) \vee M(p_2)$, the truth value of the disjunction of the fuzzy propositions.



(Analogous to Example 15, one can design mfPn, which gives the conjunction of the truth values of two fuzzy propositions.)

4. Category of Marked Fuzzy Petri Net

In this section, a category of marked fuzzy Petri net, inspired from [2], is introduced.

Definition 16. For mfPn's $(P_1, T_1, I_1^-, I_1^+, M_1)$ and $(P_2, T_2, I_2^-, I_2^+, M_2)$, a function $f : P_2 \to P_1$, (M_1, M_2) is said to be f-ok if $(M_1 \circ f)(p) \le M_2(p)$, for all $p \in P_2$.

Remark 17. MFPN shall denote the category of all mfPn's, with mfPn-morphisms $(f, g) : (N_1, M_1) \to (N_2, M_2)$ being the fPn-morphisms $(f, g) : N_1 \to N_2$ such that (M_1, M_2) is f-ok.

Proposition 18. *Let $(N_1, M_1) = (P_1, T_1, I_1^-, I_1^+, M_1)$ and $(N_2, M_2) = (P_2, T_2, I_2^-, I_2^+, M_2)$ be two mfPn's and let $(f, g) : (N_1, M_1) \to (N_2, M_2)$ be an mfPn-morphism. Then for $t_1 \in T_1$, $g(t_1)$ is enabled at M_2, if t_1 is enabled at M_1.*

Proof. As $(f, g) : N_1 \to N_2$ is an mfPn-morphism, $I_1^-(f(p_2), t_1) \ge I_2^-(p_2, g(t_1))$ and $M_2(p_2) \ge M_1(f(p_2))$, for all $(p_2, t_1) \in P_2 \times T_1$. Also, as t_1 is enabled at M_1, we have $I_1^-(p_1, t_1) \le M_1(p_1)$, for all $p_1 \in P_1$, whence $I_1^-(f(p_2), t_1) \le M_1(f(p_2))$, for all $p_2 \in P_2$. But $I_2^-(p_2, g(t_1)) \le I_1^-(f(p_2), t_1) \le M_1(f(p_2)) \le M_2(p_2)$, whereby, $I_2^-(p_2, g(t_1)) \le M_2(p_2)$, for all $p_2 \in P_2$. Thus, $g(t_1)$ is enabled at M_2. □

Proposition 19. *Let $(N_1, M_1) = (P_1, T_1, I_1^-, I_1^+, M_1)$ and $(N_2, M_2) = (P_2, T_2, I_2^-, I_2^+, M_2)$ be two mfPn's and $(f, g) : (N_1, M_1) \to (N_2, M_2)$ be an mfPn-morphism. Then for $t_1 \in T_1$, $((M_1)_{t_1}, (M_2)_{g(t_1)})$ is f-ok, if $M_1 \xrightarrow{t_1} (M_1)_{t_1}$.*

Proof. From the above proposition, it is clear that $M_2 \xrightarrow{g(t_1)} (M_2)_{g(t_1)}$. Also, as $(f, g) : (N_1, M_1) \to (N_2, M_2)$ is an mfPn-morphism, $I_1^-(f(p_2, t_1)) \ge I_2^-(p_2, g(t_1))$, $I_1^+(f(p_2, t_1)) \le I_2^+(p_2, g(t_1))$, and $M_1(f(p_2)) \le M_2(p_2)$, for all $p_2 \in P_2$. Consequently, for all $(p_2, t_1) \in P_2 \times T_1, M_1(f(p_2)) - I_1^-(f(p_2), t_1) + I_1^+(f(p_2), t_1) \le M_2(p_2) - I_2^-(p_2, g(t_1)) + I_2^+(p_2, g(t_1))$, whereby, $\min\{1, M_1(f(p_2)) - I_1^-(f(p_2), t_1) + I_1^+(f(p_2), t_1)\} \le \min\{1, (M_2(p_2) - I_2^-(p_2, g(t_1)) + I_2^+(p_2, g(t_1))\}$. Thus, $(M_1)_{t_1}(f(p_2)) \le (M_2)_{g(t_1)}(p_2)$ for all $(p_2, t_1) \in P_2 \times T_1$. Hence $((M_1)_{t_1}, (M_2)_{g(t_1)})$ is f-ok, for $t_1 \in T_1$. □

Definition 20. The *product* of two mfPn's (N_1, M_1) and (N_2, M_2) is the mfPn $(N_1 \times N_2, M_1 \oplus M_2)$, where $N_1 \times N_2$ is the product of fPn's N_1 and N_2 and $M_1 \oplus M_2 : \widetilde{P}_1 \cup \widetilde{P}_2 \to [0, 1]$ is given by

$$(M_1 \oplus M_2)(p, n) = \begin{cases} M_1(p) & \text{if } n = 0, \\ M_2(p) & \text{if } n = 1, \end{cases} \tag{9}$$

for all $(p, n) \in \widetilde{P}_1 \cup \widetilde{P}_2$.

Proposition 21. *The FPN-morphisms $(\rho_i, \pi_i) : N_1 \times N_2 \to N_i$, given in Proposition 5 are MFPN-morphisms from $(N_1 \times N_2, M_1 \oplus M_2)$ to (N_i, M_i), $i = 1, 2$.*

Proof. Since $(M_1 \oplus M_2)(\rho_1((p, n))) = (M_1 \oplus M_2)((p, 0)) = M_1(p)$ and $(M_1 \oplus M_2)(\rho_2((p, n))) = (M_1 \oplus M_2)((p, 1)) = M_2(p)$, $(M_1 \oplus M_2, M_1)$ and $(M_1 \oplus M_2, M_2)$ are ρ_1-ok and ρ_2-ok, respectively. Hence $(\rho_i, \pi_i), i = 1, 2$, are MFPN-morphisms. □

Using Propositions 5 and 21, the next proposition is evident.

Proposition 22. *The product of mfPn's is the categorical product in MFPN.*

Definition 23. The *coproduct* of two mfPn's (N_1, M_1) and (N_2, M_2) is the mfPn $(N_1 \oplus N_2, M_1 \vee M_2)$, where $N_1 \oplus N_2$ is the coproduct of fPn's N_1 and N_2 and $M_1 \vee M_2 : P_1 \times P_2 \to [0, 1]$ is given by $(M_1 \vee M_2)(p_1, p_2) = M_1(p_1) \vee M_2(p_2)$, for all $(p_1, p_2) \in P_1 \times P_2$.

Similar to Propositions 21 and 22, the following two propositions can also be proved.

Proposition 24. *The FPN-morphisms $(\pi_i, \rho_i) : N_i \to N_1 \oplus N_2, i = 1, 2,$ given in Proposition 8 are MFPN-morphisms from (N_i, M_i) to $(N_1 \oplus N_2, M_1 \vee M_2)$.*

Proposition 25. *The coproduct of mfPn's is the categorical coproduct in MFPN.*

5. Relationship between FPN and MFPN

There is an obvious functor $k_1 : MFPN \to FPN$, given by $(N, M) \mapsto N$ and $(f, g) \mapsto (f, g)$.

We omit the easy verification of the following observations.

Proposition 26. *There are full and faithful functors* k_2, k_3 : **FPN** $\rightarrow$ **MFPN**, *which, on objects, are respectively, given by* $N \mapsto (N, \mathbf{0})$ *and* $N \mapsto (N, \mathbf{1})$, *where* $\mathbf{0}$ *and* $\mathbf{1}$, *are respectively, the 0-valued and the 1-valued constant functions, and which leave the morphisms unchanged.*

It is easy to prove the following.

Proposition 27. *The functor* k_2 *(resp.,* k_3*) is left adjoint (resp., right adjoint) to the functor* k_1.

Thus, we have the following.

Proposition 28. *The category FPN is isomorphic to a full reflective subcategory, and also to a full coreflective subcategory, of MFPN.*

6. Conclusion

We note that nothing has been said about the symmetric monoidal closed structure of the category MFPN of marked fuzzy Petri nets. An obvious attempt to make MFPN symmetric monoidal closed would appear to be as follows. Given mfPn's (N_0, M_0) and (N_1, M_1), the fPn's $N_1 \otimes N_0$ and $N_1^{N_0}$ (cf. Definition 10) can be made mfPn's by taking their respective fuzzy markings to be $M_1 \otimes M_0 : P_1^{T_0} \times P_0^{T_1} \rightarrow [0,1]$ and $M_1^{M_0} : T_0 \times P_1 \rightarrow [0,1]$, defined as $M_1 \otimes M_0((\lambda, \mu)) = \vee\{\vee_{T_0}\{M_1(\lambda(t_0))\}, \vee_{T_1}\{M_0(\mu(t_1))\}\}$ and $M_1^{M_0}((t_0, p_1)) = M_1(p_1)$, for all $(\lambda, \mu) \in P_1^{T_0} \times P_0^{T_1}$, for all $(t_0, p_1) \in T_0 \times P_1$. However, for each fixed mfPn (N_0, M_0), the resulting functors $- \otimes (N_0, M_0)$, $(-)^{(N_0, M_0)} : MFPN \rightarrow MFPN$ do not turn out to be adjoint. As an attempt to repair the above situation, $M_1 \otimes M_0$ may be redefined as $M_1 \otimes M_0((\lambda, \mu)) = \vee_{T_0}\{M_1(\lambda(t_0))\}$, so that the functor $(-)^{(N_0, M_0)}$ does, now, turn out to be right adjoint to the (modified) functor $- \otimes (N_0, M_0)$. However, the symmetry of $\otimes$ in this modified setup is now lost (this situation is similar to the one noted in [2]). So there may be a different symmetric monoidal closed structure on MFPN which we have not been able to find presently.

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