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Research Article

Spatial Analysis and Fuzzy Relation Equations

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We implement an algorithm that uses a system of fuzzy relation equations (SFRE) with the max-min composition for solving a problem of spatial analysis. We integrate this algorithm in a Geographical Information System (GIS) tool, and the geographical area under study is divided in homogeneous subzones (with respect to the parameters involved) to which we apply our process to determine the symptoms after that an expert sets the SFRE with the values of the impact coefficients. We find that the best solutions and the related results are associated to each subzone. Among others, we define an index to evaluate the reliability of the results.

1. Introduction

A Geographical Information System (GIS) is used as a support decision system for problems in a spatial domain; in many cases, we use a GIS to analyze spatial distribution of data, spatial relations, the impact of event data on spatial areas; simple examples of this analysis are the creation of thematic maps, the geoprocessing operators, the buffer analysis, and so forth. Often the expert analyzes spatial data in a decision making process with the help of a GIS which involves integration of images, spatial layers, attributes information and an inference mechanism based on these attributes. The diversity and the inhomogeneity between the individual layers of spatial information and the inaccuracy of the results can lead to uncertain decisions, so that one needs the use of fuzzy inference calculus to handle these uncertain information. Many authors [1–5] propose models to solve spatial problems based on fuzzy relational calculus. In this paper, we propose an inferential method to solve spatial problems based on an algorithm for the resolution of a system of fuzzy relation equations (shortly, SFRE) given in [6] (cf. also [7, 8]) and applied in [9] to solve industrial application problems. Here we integrate this algorithm in the context of a GIS architecture. Usually an SFRE with max-min composition is read as

$$\begin{gathered}(a_{11} \wedge x_1) \vee \cdots \vee (a_{1n} \wedge x_n) = b_1,\\ (a_{21} \wedge x_1) \vee \cdots \vee (a_{2n} \wedge x_n) = b_2,\\ \vdots\\ (a_{m1} \wedge x_1) \vee \cdots \vee (a_{mn} \wedge x_n) = b_m.\end{gathered} \tag{1}$$

The system (1) is said consistent if it has solutions. In his pioneering paper [10], the author determines the greatest solution in case of max-min composition. After these results, many researchers have found algorithms which determine minimal solutions of max-min fuzzy relation equations (cf. e.g., [11–18]). In [6, 7] a method is described for the consistence of the system (1), and moreover it calculates the complete set of the solutions. This method is schematized in Figure 1 and described below.

(i) Input extraction: the input data are extracted and stored in the dataset.

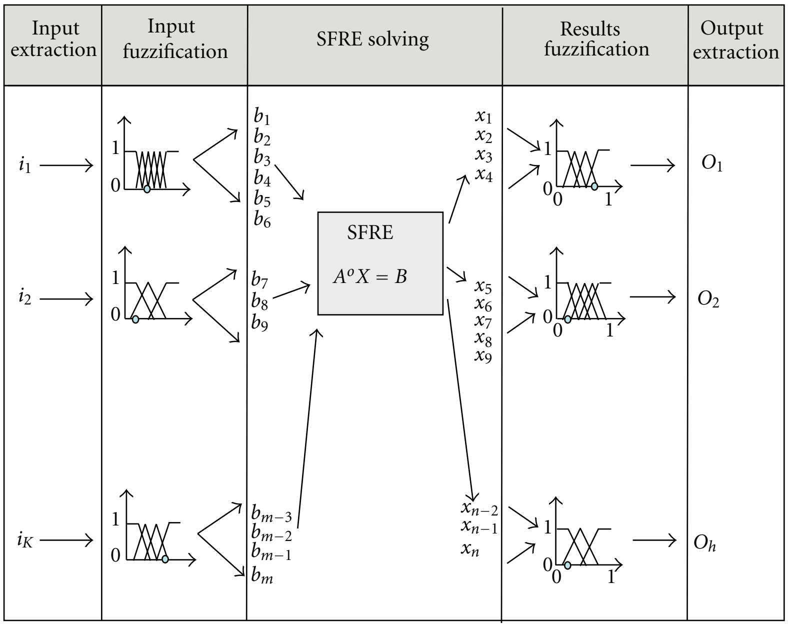

Figure 1: Resolution process of an SFRE.

(ii) The input variable is fuzzified. A fuzzy partition of the input domain is created; the corresponding membership degree of every input data is assigned to each fuzzy set.

(iii) The membership degrees of each fuzzy set determine the coefficients $\{b_1, \ldots, b_m\}$ of (1). The values of the coefficients a_{ij} are set by the expert and the whole set of solutions $(x_1, \ldots, x_n)$ of (1) is determined as well.

(iv) A fuzzy partition of the domain $[0, 1]$ is created for the output variables $o_1, \ldots, o_k$; every fuzzy set of the partition corresponds to a determined value x_j.

(v) The output data $o_1, \ldots, o_k$ are extracted. A partition of fuzzy sets corresponds to each output variable o_j $(j = 1, \ldots, k)$; in this phase the linguistic label of the most appropriate fuzzy set is assigned to the output variable o_j.

This process has been applied to a real spatial problem in which the input data vary for each subzone of the geographical area. We have the same input data, and the expert applies the same SFRE (1) on each subzone. The expert starts from a valuation of input data, and he uses linguistic labels for the determination of the output results for each subzone. The input data are the facts or symptoms; the parameters to be determined are the causes. For example, let us consider a planning problem. A city planner needs to determine in each subzone the mean state of buildings (x_1) and the mean soil permeability (x_2), knowing the number of collapsed building in the last year (b_1) and the number of flooding in the last year (b_2). In Figure 2, we suppose to create for each symptom's and cause's variable domain a fuzzy partition of three fuzzy sets (generally, one is faced with trapezoidal or triangular fuzzy number, this last one is denoted in the sequel shortly with the acronym TFN). The expert creates the SFRE (1) for each subzone by setting the impact matrix A, whose entries a_{ij} $(i = 1, \ldots, n$ and $j = 1, \ldots, m)$ represent the impact of the jth cause x_j to the production of the ith symptom b_i, where the value of b_i is the membership degree in the corresponding fuzzy set and let $B = [b_1, \ldots, b_m]$. In another subzone the input data vector B and the matrix A can vary. For example, we consider the equation:

$$(0.8 \wedge x_1) \vee (0.2 \wedge x_2) \vee (0.0 \wedge x_3) \vee (0.8 \wedge x_4) \vee (0.3 \wedge x_5) \vee (0.0 \wedge x_6) = b_3 = 0.9. \tag{2}$$

The expert sets for the symptom b_3 = "collapsed building in the last year = high" = 0.9, an impact 0.8 of the variable "mean state of buildings = scanty", an impact 0.2 of the variable "mean state of buildings = medium", an impact 0.0 of the variable "mean state of buildings = high", an impact 0.8 of the variable "mean soil permeability = low", an impact 0.3 of the variable "mean soil permeability = medium", or an impact 0.0 of the variable "mean soil permeability = high".

We can determine the maximal interval solutions of (1). Each maximal interval solution is an interval whose extremes are the values taken from a minimal solution and from the greatest solution. Every value x_i belongs to this interval. If the SFRE (1) is inconsistent, it is possible to determine the rows for which no solution is permitted. If the expert decides to exclude the row for which no solution is permitted, he considers that the symptom b_i (for that row) is not relevant to its analysis, and it is not taken into account. Otherwise, the expert can modify the setting of the coefficients of the matrix A to verify if the new system has some solution. In general, the SFRE (1) has T

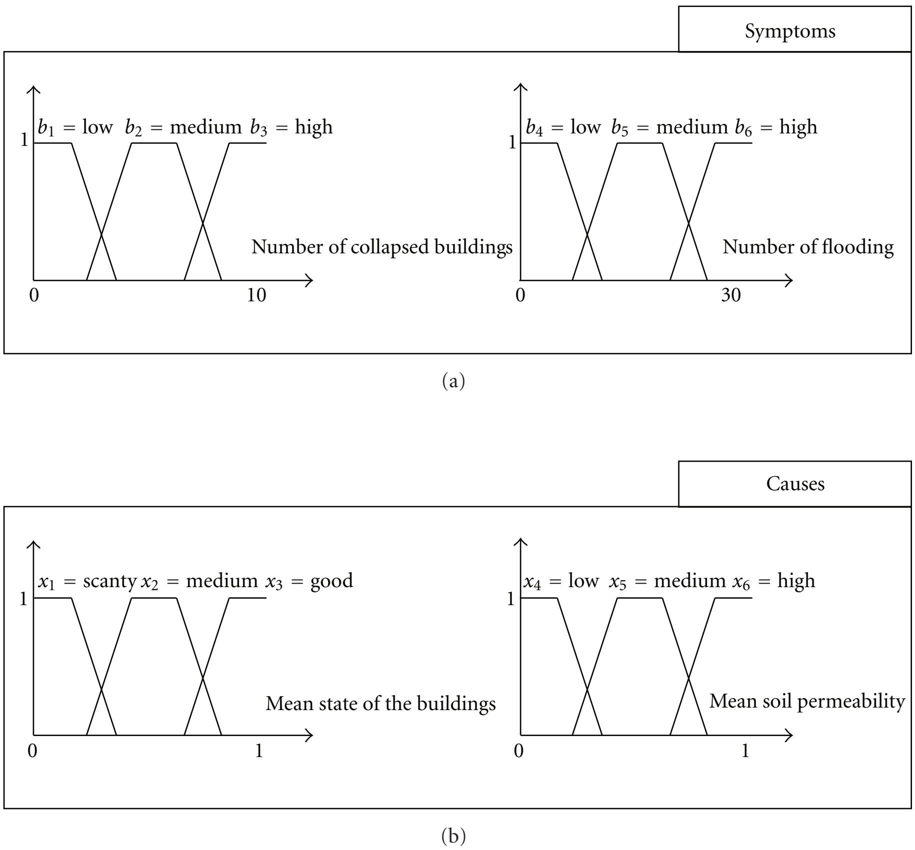

Figure 2: Examples of trapezoidal fuzzy numbers used for symptoms and causes.

maximal interval solutions $X_{\max(1)},\dots,X_{\max(T)}$. In order to describe the extraction process of the solutions, let $X_{\max(t)}$, $t \in \{1,\dots,T\}$, be a maximal interval solution given below, where X^{low} is a minimal solution and X^{gr} is the greatest solution. Our aim is to assign the linguistic label of the most appropriate fuzzy sets corresponding to the unknown $\{x_{j_1}, x_{j_1},\dots,x_{j_s}\}$ related to an output variable o_s, $s = 1,\dots,k$. For example, assume that the three fuzzy sets x_1, x_2, x_3 (resp., x_4, x_5, x_6) are related to o_1 (resp., o_2) and are represented from the TFNs given in Table 1, where $\text{INF}(j)$, $\text{MEAN}(j)$, and $\text{SUP}(j)$ are the three fundamental values of the generic TFN x_j, $j = j_1, \dots, j_s$. We can write their membership functions $\mu_{j_1}, \mu_{j_2},\dots,\mu_{j_h}$ as follows:

$$\mu_{j_1} = \begin{cases} 1, & \text{if } \text{INF}(j_1) \le x \le \text{MEAN}(j_1), \\ \dfrac{\text{SUP}(j_1) - x}{\text{SUP}(j_1) - \text{MEAN}(j_1)}, & \text{if } \text{MEAN}(j_1) < x \le \text{SUP}(j_1), \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

$$\mu_j = \begin{cases} \dfrac{x - \text{INF}(j)}{\text{MEAN}(j) - \text{INF}(j)}, & \text{if } \text{INF}(j) \le x \le \text{MEAN}(j), \\ \dfrac{\text{SUP}(j) - x}{\text{SUP}(j) - \text{MEAN}(j)}, & \text{if } \text{MEAN}(j) < x \le \text{SUP}(j), \\ 0, & \text{otherwise}, \end{cases} \quad j \in \{j_2,\dots,j_{s-1}\}, \tag{4}$$

$$\mu_{j_s} = \begin{cases} \dfrac{x - \text{INF}(j_s)}{\text{MEAN}(j_s) - \text{INF}(j_s)}, & \text{if } \text{INF}(j_s) \le x \le \text{MEAN}(j_s), \\ 1, & \text{if } \text{MEAN}(j_s) < x \le \text{SUP}(j_s), \\ 0, & \text{otherwise}. \end{cases} \tag{5}$$

If $X\text{Min}_t(j)$ (resp., $X\text{Max}_t(j)$) is the min (resp., max) value of every interval corresponding to the unknown x_j, we can calculate the arithmetical mean value $X\text{Mean}_t(j)$ of

Table 1: TFNs values for the fuzzy sets.

Unknown	INF(j)	MEAN(j)	SUP(j)
x_1	0.0	0.2	0.4
x_2	0.3	0.5	0.7
x_3	0.6	0.8	1.0
x_4	0.0	0.2	0.4
x_5	0.3	0.5	0.7
x_6	0.6	0.8	0.1

the jth component of the above maximal interval solution $X_{\max(t)}$ as

$$X\mathrm{Mean}_t(j) = \frac{X\mathrm{Min}_t(j) + X\mathrm{Max}_t(j)}{2}, \tag{6}$$

and we get the vector column $X\mathrm{Mean}_t = [X\mathrm{Mean}_t(1), \dots, X\mathrm{Mean}_t(n)]^{-1}$ (cf. Table 2). The value given from $\max\{X\mathrm{Mean}_t(j_1), \dots, X\mathrm{Mean}_t(j_s)\}$ obtained for the unknowns $x_{j_1}, \dots, x_{j_s}$ corresponding to the output variable o_s, is the linguistic label of the fuzzy set assigned to o_s and it is denoted by $\mathrm{score}_t\ (o_s)$, defined also as reliability of o_s in the interval solution t. In our example, we have that "o_1 = *mean state of buildings* = *scanty*" and "o_2 = *mean soil permeability* = *medium*", hence $\mathrm{score}_t(o_1) = 0.70$ and $\mathrm{score}_t(o_2) = 0.55$. For the output vector $O = [o_1, \dots, o_k]$, we define the following reliability index in the interval solution t as

$$\mathrm{Rel}_t(O) = \frac{1}{k} \cdot \sum_{s=1}^{k} \mathrm{score}_t(o_s) \tag{7}$$

and then as final reliability index of O, the number $\mathrm{Rel}(O) = \max\{\mathrm{Rel}_t(O) : t = 1, \dots, T\}$.

In our example, we have $\mathrm{Rel}_t(O) = (0.7 + 0.55)/2 = 0.625$. Therefore, the higher the reliability of our solution, the closer the final reliability index $\mathrm{Rel}(O)$ to 1. In Section 2, we give an extended and articulated overview on how to determine the whole set of the solutions of an SFRE, and in Section 3 we show how the proposed algorithm is applied in spatial analysis. Section 4 contains the results of our simulation.

2. SFRE: An Extended Overview

In this paper, we investigate the solutions of the SFRE (1), which is abbreviated in the following known form:

$$A \circ X = B, \tag{8}$$

where $A = (a_{ij})$ is the matrix of coefficients, $X = (x_1, x_2, \dots, x_n)^{-1}$ is the column vector of the unknowns, and $B = (b_1, b_2, \dots, b_m)^{-1}$ is the column vector of the known terms, being $a_{ij}, x_j, b_i \in [0, 1]$ for each $i = 1, \dots, m$ and $j = 1, \dots, n$. We have the following definitions and terminologies: the whole set of all solutions X of the SFRE (8) is denoted by Ω. If $\Omega \neq \emptyset$, then the SFRE (8) is called consistent, otherwise it is called inconsistent. A solution $\widehat{X} \in \Omega$ is called a minimal solution if $X \leq \widehat{X}$ for some $X \in \Omega$ implies $X = \widehat{X}$, where "$\leq$" is the partial order induced in Ω from the natural order of $[0, 1]$. If the minimal solution is unique, then it is the least (or minimum) solution of the SFRE (8). We also recall that the system (8) has the unique greatest (or maximum) solution $X^{\mathrm{gr}} = (x_1^{\mathrm{gr}}, x_2^{\mathrm{gr}}, \dots, x_n^{\mathrm{gr}})^{-1}$ if $\Omega \neq \emptyset$ [10]. A matrix interval X_{interval} of the following type:

$$X_{\mathrm{interval}} = \begin{pmatrix} [a_1, b_1] \\ [a_2, b_2] \\ [\dots, \dots] \\ [a_n, b_n] \end{pmatrix}, \tag{9}$$

where $[a_j, b_j] \subseteq [0, 1]$ for each $j = 1, \dots, n$, is called an interval solution of the SFRE (8) if every $X = (x_1, x_2, \dots, x_n)^{-1}$ such that $x_j \in [a_j, b_j]$ for each $j = 1, \dots, n$, belongs to Ω. If a_j is a membership value of a minimal solution and b_j is a membership value of X^{gr} for each $j = 1, \dots, n$, then X_{interval} is called a maximal interval solution of the SFRE (8), and it is denoted by $X_{\max(t)}$, where t varies from 1 to the number of minimal solutions. The SFRE (8) is said to be in normal form if $b_1 \geq b_2 \geq \cdots \geq b_m$. The time computational complexity to reduce an SFRE in a normal form is polynomial [6, 8]. Now we consider the matrix $A^* = (a_{ij}^*)$ so defined:

$$a_{ij}^* = \begin{cases} 0, & \text{if } a_{ij} < b_i, \\ b_i, & \text{if } a_{ij} = b_i, \\ 1, & \text{if } a_{ij} > b_i, \end{cases} \tag{10}$$

where $i = 1, \dots, m$ and $j = 1, \dots, n$. The linguistic description of a_{ij}^* as S-type coefficient (Smaller) if $a_{ij} < b_i$, E-type coefficient (Equal) if $a_{ij} = b_i$, and G-type coefficient (Greater) if $a_{ij} > b_i$ is often used. A^* is called augmented matrix, and the system $A^* \circ X = B$ is said associated to the SFRE (8). Without loss of generality, from now on we suppose that the system (8) is in normal form. We also obtained the following definitions and results from [6, 8, 19, 20].

Definition 1. Let the SFRE (8) be consistent and $A_j^* = \{a_{1j}^*, \dots, a_{mj}^*\}$. If $\mathrm{A}^*(j)$ contains G-type coefficients and $k \in \{1, \dots, m\}$ is the greatest index of row such that $a_{kj}^* = 1$, then the following coefficients in $\mathrm{A}^*(j)$ are called selected:

(i) a_{ij}^* for $i \in \{1, \dots, k\}$ with $a_{ij}^* \geq b_i = b_k$,

(ii) a_{ij}^* for $i \in \{k+1, \dots, m\}$ with $a_{ij}^* = b_i$.

Definition 2. If $A^*(j)$ does not contain G-type coefficients, but it contain E-type coefficients and $r \in \{1, \dots, m\}$ is the smallest index of row such that $a_{rj}^* = b_r$, then any $a_{ij}^* = b_i$ in $A^*(j)$ for $i \in \{r, \dots, m\}$ is called selected.

Theorem 3. *Consider an SFRE* (8)*. Then the following occurs.*

(i) *The SFRE* (8) *is consistent if and only if there exist at least one selected coefficient for each ith equation,* $i = 1, \dots, m$.

(ii) *The complexity time function for determining the consistency of the SFRE* (8) *is* $O(m \cdot n)$.

Table 2: TFNs mean values.

Output variable	Unknown component	Linguistic label	$XMin_t(j)$	$XMax_t(j)$	$XMean_t(j)$
	x_1	scanty	0.6	0.8	0.70
o_1	x_2	medium	0.2	0.4	0.30
	x_3	good	0.0	0.1	0.05
	x_4	low	0.3	0.5	0.40
o_2	x_5	medium	0.4	0.7	0.55
	x_6	good	0.0	0.3	0.15

Consequently, when an SFRE (8) is inconsistent, the equations for which no element is a selected coefficient could not be satisfied simultaneously with the other equations having at least one selected coefficient. Furthermore, a vector IND = (IND(1), ..., IND(m)) is defined by setting IND(i) equal to the number of selected coefficients in the ith equation for each $i = l, \ldots, m$. If IND(i) = 0, then all the coefficients in the ith equation are not selected and the system is inconsistent. The system is consistent if IND(i) $\neq 0$ for each $i = l, \ldots, m$ and the product

$$\mathrm{PN2} = \prod_{i=1}^{m} \mathrm{IND}(i), \tag{11}$$

gives the upper bound of the number of the eventual minimal solutions.

Theorem 4. *Let the SFRE* (8) *be consistent. Then the following occurs.*

(i) *The SFRE has a unique greatest solution X^{gr} with component $x_j^{gr} = b_k$ if the jth column $A^*(j)$ of A^* contains selected G-type coefficients a_{kj}^* and $x_j^{gr} = 1$ otherwise.*

(ii) *The complexity time function for computing X^{gr} is $O(m \cdot n)$.*

A help matrix $H = (h_{ij})$, with $i = 1, \ldots, m$ and $j = 1, \ldots, n$, is defined as follows:

$$h_{ij} = \begin{cases} b_i, & \text{if } a_{ij}^* \text{ is selected,} \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Let $|H_i|$ be the number of coefficients h_{ij} in the ith equation of the SFRE (8). Then the number of potential minimal solutions cannot exceed the value

$$\mathrm{PN1} = \prod_{i=1}^{m} |H_i|, \tag{13}$$

where PN2 $\leq$ PN1.

Definition 5. Let $h_i = (h_{i1}, h_{i2}, \ldots, h_{in})$ and $h_k = (h_{k1}, h_{k2}, \ldots, h_{kn})$ be the ith and the kth rows of the help matrix H. If for each $j = 1, \ldots n$, $h_{ij} \neq 0$ implies both $h_{kj} \neq 0$ and $h_{kj} \leq h_{ij}$, then the ith row (resp., equation) is said dominant over the kth row in H (resp., equation) or that the kth row (resp., equation) is said dominated by the ith row (resp., equation).

In other terms, if the ith equation is dominant over the kth equation in (8), then the kth equation is a redundant equation of the system. By using Definition 5, we can build a matrix of dimension $m \times n$, called dominance matrix, with components:

$$h_{ij}^* = \begin{cases} 0, & \text{if the } i\text{th equation is dominated by} \\ & \text{another equation,} \\ h_{ij}, & \text{otherwise.} \end{cases} \tag{14}$$

For each $i = 1, \ldots, m$, now we set $|H_i^*|$ as the number of coefficients $h_{ij}^* = b_i \neq 0$ in the ith row of the dominance matrix H^*. When this value is 0, we set $|H_i^*| = 1$. Then the number of potential minimal solutions of the SFRE cannot exceed the value

$$\mathrm{PN3} = \prod_{i=1}^{m} |H_i^*|, \tag{15}$$

where PN3 $\leq$ PN2 $\leq$ PN1. In [6, 8, 20], the authors use the symbol $\langle b_i/j \rangle$ to indicate the coefficients $h_{ij}^* = b_i \neq 0$. We have $h_{ij}^* \wedge x_j = b_i$ if $x_j \in [b_i, 1]$ and $x_j = b_i$ is the jth component of a minimal solution. A solution of the ith equation can be written as

$$H_i = \sum_{j=1}^{n} \left\langle \frac{b_i}{j} \right\rangle. \tag{16}$$

In [6, 8] the concept of concatenation W is introduced to determine all the components of the minimal solutions and it is given by

$$W = \prod_{i=1}^{m} H_i = \prod_{i=1}^{m} \left(\sum_{j=1}^{n} \left\langle \frac{b_i}{j} \right\rangle \right). \tag{17}$$

The following properties hold:

(i) commutativity:

$$\left\langle \frac{b_{i_1}}{j_1} \right\rangle \left\langle \frac{b_{i_2}}{j_2} \right\rangle = \left\langle \frac{b_{i_2}}{j_2} \right\rangle \left\langle \frac{b_{i_1}}{j_1} \right\rangle, \tag{18}$$

(ii) associativity:

$$\left\langle \frac{b_{i_1}}{j_1} \right\rangle \left(\left\langle \frac{b_{i_2}}{j_2} \right\rangle \left\langle \frac{b_{i_3}}{j_3} \right\rangle \right) = \left(\left\langle \frac{b_{i_1}}{j_1} \right\rangle \left\langle \frac{b_{i_2}}{j_2} \right\rangle \right) \left\langle \frac{b_{i_3}}{j_3} \right\rangle, \tag{19}$$

(iii) distributivity with respect to the addition:

$$\left\langle \frac{b_{i_1}}{j_1} \right\rangle \left(\left\langle \frac{b_{i_2}}{j_2} \right\rangle + \left\langle \frac{b_{i_3}}{j_3} \right\rangle \right) = \left\langle \frac{b_{i_1}}{j_1} \right\rangle \left\langle \frac{b_{i_2}}{j_2} \right\rangle + \left\langle \frac{b_{i_1}}{j_1} \right\rangle \left\langle \frac{b_{i_3}}{j_3} \right\rangle, \quad (20)$$

(iv) absorption for multiplication:

$$\left\langle \frac{b_{i_1}}{j_1} \right\rangle \left\langle \frac{b_{i_2}}{j_2} \right\rangle = \begin{cases} \left\langle \dfrac{b_{i_1} \wedge b_{i_2}}{j} \right\rangle, & \text{if } j_1 = j_2 = j, \\ \text{unchanged}, & \text{otherwise}, \end{cases} \quad (21)$$

(v) absorption for addition:

$$\begin{aligned} &\left\langle \frac{b_{i_1}}{j_1} \right\rangle \left\langle \frac{b_{i_2}}{j_2} \right\rangle \cdots \left\langle \frac{b_{i_m}}{j_n} \right\rangle + \left\langle \frac{b_{k_1}}{j_1} \right\rangle \left\langle \frac{b_{k_2}}{j_2} \right\rangle \cdots \left\langle \frac{b_{k_m}}{j_n} \right\rangle \\ &= \begin{cases} \left\langle \dfrac{b_{i_1}}{j_1} \right\rangle \left\langle \dfrac{b_{i_2}}{j_2} \right\rangle \cdots \left\langle \dfrac{b_{i_m}}{j_n} \right\rangle, & \text{if } b_{i_h} = b_{k_h}, \\ & h \in \{1, \ldots, m\}, \\ \text{unchanged}, & \text{otherwise}. \end{cases} \end{aligned} \quad (22)$$

We can determine the minimal solutions $X^{\text{low}(t)} = (x_1^{\text{low}(t)}, x_2^{\text{low}(t)}, \ldots, x_n^{\text{low}(t)})^{-1}$, $t \in \{1, \ldots, \text{PN}(3)\}$, with components

$$x_j^{\text{low}(t)} = \begin{cases} b_{i_t}, & \text{if } b_{i_t} \neq 0, \\ 0, & \text{otherwise}. \end{cases} \quad (23)$$

The above definitions shall be clarified in the following example of an SFRE with 4 equations and 6 unknown:

$$\begin{aligned} &(1.0 \wedge x_1) \vee (0.0 \wedge x_2) \vee (0.0 \wedge x_3) \\ &\quad \vee (0.9 \wedge x_4) \vee (0.2 \wedge x_5) \vee (0.0 \wedge x_6) = 0.1, \\ &(0.5 \wedge x_1) \vee (0.3 \wedge x_2) \vee (0.4 \wedge x_3) \\ &\quad \vee (0.5 \wedge x_4) \vee (0.3 \wedge x_5) \vee (0.4 \wedge x_6) = 0.3, \\ &(0.7 \wedge x_1) \vee (0.4 \wedge x_2) \vee (0.2 \wedge x_3) \\ &\quad \vee (0.7 \wedge x_4) \vee (0.4 \wedge x_5) \vee (0.2 \wedge x_6) = 0.3, \\ &(0.4 \wedge x_1) \vee (0.7 \wedge x_2) \vee (0.2 \wedge x_3) \\ &\quad \vee (0.4 \wedge x_4) \vee (0.7 \wedge x_5) \vee (0.2 \wedge x_6) = 0.3. \end{aligned} \quad (24)$$

We have

$$A = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.9 & 0.2 & 0.0 \\ 0.5 & 0.3 & 0.4 & 0.5 & 0.3 & 0.4 \\ 0.7 & 0.4 & 0.2 & 0.7 & 0.4 & 0.2 \\ 0.4 & 0.7 & 0.2 & 0.4 & 0.7 & 0.2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0.1 \\ 0.3 \\ 0.3 \\ 0.3 \end{pmatrix}. \quad (25)$$

By using the normal form, we obtain that

$$A = \begin{pmatrix} 0.5 & 0.3 & 0.4 & 0.5 & 0.3 & 0.4 \\ 0.7 & 0.4 & 0.2 & 0.7 & 0.4 & 0.2 \\ 0.4 & 0.7 & 0.2 & 0.4 & 0.7 & 0.2 \\ 1.0 & 0.0 & 0.0 & 0.9 & 0.2 & 0.0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.3 \\ 0.1 \end{pmatrix}. \quad (26)$$

Now we compute the matrix A^* and the vector IND as follows:

$$A^* = \begin{pmatrix} 1.0 & 0.3 & 1.0 & 1.0 & 0.3 & 1.0 \\ 1.0 & 1.0 & 0.0 & 1.0 & 1.0 & 0.0 \\ 1.0 & 1.0 & 0.0 & 1.0 & 1.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 1.0 & 1.0 & 0.0 \end{pmatrix}, \qquad \text{IND} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 3 \end{pmatrix}. \quad (27)$$

The SFRE is consistent because each component of IND is not null. The greatest solution is given by

$$X^{\text{gr}} = \begin{pmatrix} 0.1 \\ 0.3 \\ 0.3 \\ 0.1 \\ 0.1 \\ 0.3 \end{pmatrix}. \quad (28)$$

Now we calculate the help matrix H and the dominant matrix H^* as follows:

$$\begin{aligned} H &= \begin{pmatrix} 0.0 & 0.3 & 0.3 & 0.0 & 0.0 & 0.3 \\ 0.0 & 0.3 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.0 & 0.0 & 0.1 & 0.1 & 0.0 \end{pmatrix}, \\ H^* &= \begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.0 & 0.0 & 0.1 & 0.1 & 0.0 \end{pmatrix}. \end{aligned} \quad (29)$$

Then we have $|H_1^*| = |H_2^*| = |H_3^*| = 1$, $|H_1^*| = 3$ and hence $\text{PN3} = 3$. By using the properties (18)–(23), we have that

$$\begin{aligned} W &= \left\langle \frac{0.3}{2} \right\rangle \left(\left\langle \frac{0.1}{1} \right\rangle + \left\langle \frac{0.1}{4} \right\rangle + \left\langle \frac{0.1}{5} \right\rangle \right) \\ &= \left\langle \frac{0.1}{1} \right\rangle \left\langle \frac{0.3}{2} \right\rangle + \left\langle \frac{0.3}{2} \right\rangle \left\langle \frac{0.1}{4} \right\rangle \\ &\quad + \left\langle \frac{0.3}{2} \right\rangle \left\langle \frac{0.1}{5} \right\rangle. \end{aligned} \quad (30)$$

The three minimal solutions are given by

$$X^{\text{low}(1)} = \begin{pmatrix} 0.1 \\ 0.3 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}, \quad X^{\text{low}(2)} = \begin{pmatrix} 0.0 \\ 0.3 \\ 0.0 \\ 0.1 \\ 0.0 \\ 0.0 \end{pmatrix}, \quad X^{\text{low}(3)} = \begin{pmatrix} 0.0 \\ 0.3 \\ 0.0 \\ 0.0 \\ 0.1 \\ 0.0 \end{pmatrix}. \tag{31}$$

The three maximal interval solutions are given by

$$X_{\max(1)} = \begin{pmatrix} [0.1, 0.1] \\ [0.3, 0.3] \\ [0.0, 0.3] \\ [0.0, 0.1] \\ [0.0, 0.1] \\ [0.0, 0.3] \end{pmatrix}, \quad X_{\max(2)} = \begin{pmatrix} [0.0, 0.1] \\ [0.3, 0.3] \\ [0.0, 0.3] \\ [0.1, 0.1] \\ [0.0, 0.1] \\ [0.0, 0.3] \end{pmatrix}, \tag{32}$$

$$X_{\max(3)} = \begin{pmatrix} [0.0, 0.1] \\ [0.3, 0.3] \\ [0.0, 0.3] \\ [0.0, 0.1] \\ [0.1, 0.1] \\ [0.0, 0.3] \end{pmatrix}.$$

In order to determine if an SFRE is consistent, hence its greatest solution and minimal solutions, we have used the universal algorithm of [6, 8] based on the above concepts. For brevity of presentation, here we do not give this algorithm which has been implemented and tested under C++ language. The C++ library has been integrated in the ESRI ArcObject Library of the tool ArcGIS 9.3 for a problem of spatial analysis illustrated in Section 3.

3. SFRE in Spatial Analysis

We consider a specific area of study on the geographical map on which we have a spatial data set of "causes" and we want to analyze the possible "symptoms". We divide this area in P subzones (see, e.g, Figure 3), where a subzone is an area in which the same symptoms are derived by input data or facts, and the impact of a symptom on a cause is the same one as well. It is important to note that even if two subzones have the same input data, they can have different impact degrees of symptoms on the causes. For example, the cause that measures the occurrence of floods may be due to different degrees of importance to the presence of low porous soils or to areas subjected to continuous rains. Afterwards the area of study is divided in homogeneous subzones, hence the expert creates a fuzzy partition for the domain of each input variable and, for each subzone, he determines the values of the symptoms b_i, as the membership degrees of the corresponding fuzzy sets (cf. input fuzzification process of Figure 1). For each subzone, then the expert sets the most significant equations and the values a_{ij} of impact of the jth cause to the ith symptom creating the SFRE (1). After the determination of the set of maximal interval solutions by using the algorithm of Section 2, the expert for each interval solution calculates, for each unknown x_j, the mean interval solution $X\text{Max}_{M,t}(j)$ with (6). The linguistic label $\text{Rel}_t(o_s)$ is assigned to the output variable o_s. Then he calculates the reliability index $\text{Rel}_t(O)$, given from formula (7), associated to this maximal interval solution t. After the iteration of this step, the expert determines the reliability index (7) for each maximal interval solution, by choosing the output vector O for which $\text{Rel}(O)$ assumes the maximum value. Iterating the process for all the subzones, the expert can show the thematic map of each output variable. We schematize the whole process in Figure 4.

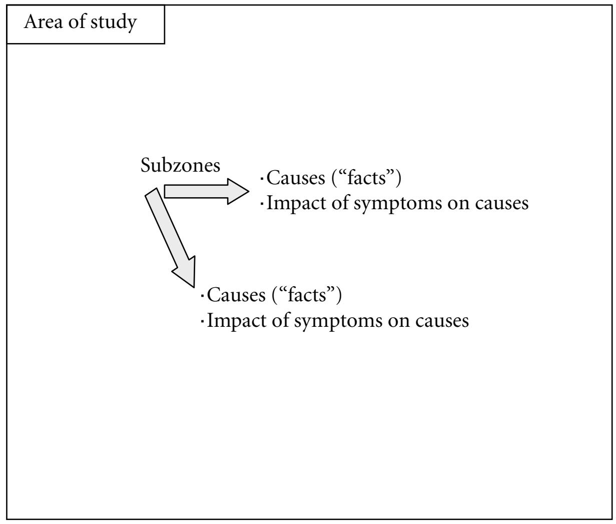

Figure 3: Subdivision in homogeneous subzones.

We suppose to subdivide the area of study in P subzones. The steps of the process are described below.

(i) In the spatial dataset, we associate k facts $i_1, \ldots, i_h$ to every subzone.

(ii) For each input fact, a fuzzy partition in m_f fuzzy sets is created for every $f = 1, \ldots, h$. To each fuzzy set, the expert associates a linguistic label. After the fuzzification process, the expert determines the m most significant equations, where $m \le m_1 + m_2 + \cdots + m_k$. The input vector $B = [b_1, \ldots, b_m]$ is set, where each component b_i ($i = 1, \ldots, m$) is the membership degree to the ith fuzzy set of the corresponding input fact. To create the fuzzy partitions, we use TFNs (cf. formulae (3), (4), (5)). The expert sets the impact of the m symptoms to the n causes by defining the impact matrix A with entries a_{ij} with $i = 1, \ldots, m$, $j = 1, \ldots, n$.

(iii) An SFRE (1) with m equations and n unknowns is created. We use the algorithm from [8] to determine

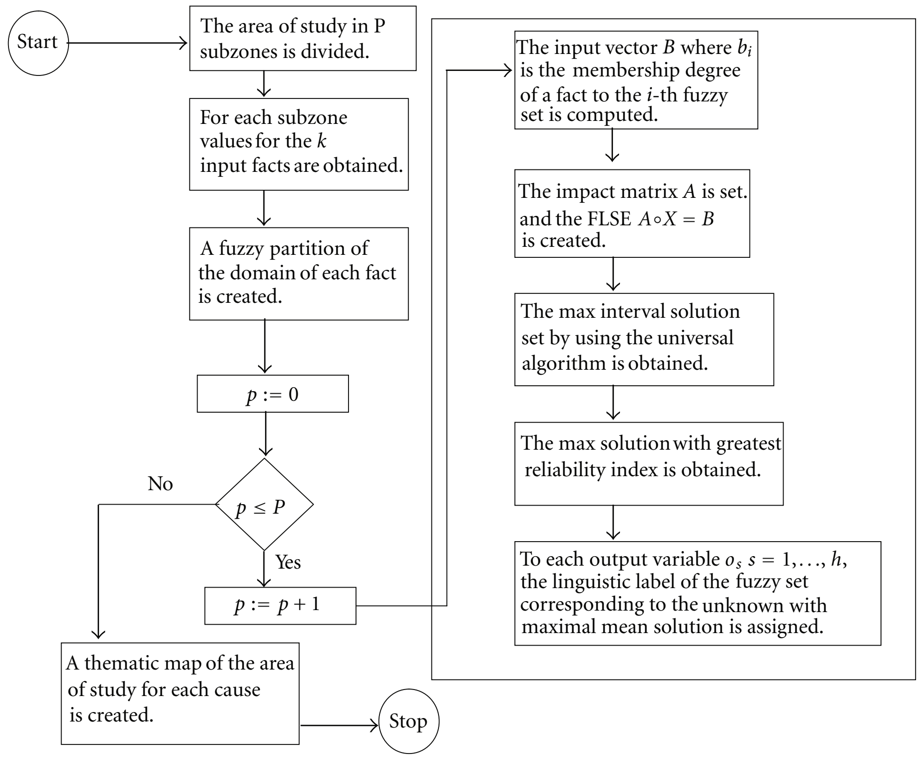

Figure 4: Flux diagram of the resolution problem.

all the solutions of (1). Thus we determine T maximal interval solutions.

(iv) $\max \mathrm{Rel}_t := 0$// (the maximal reliability is initialized to 0).

(v) For each maximal interval solution $X_{\max,t}$, with $t = 1, \ldots, T$, we define the vector column $X\mathrm{Mean}_t$ via formula (6).

(vi) $\mathrm{Rel}_t := 0$.

(vii) For each output variable o_s, with $s = 1, \ldots, k$, if $x_{j_1}, \ldots, x_{j_s}$ are the unknown associated to o_s, let $\mathrm{score}_t(s) = \max\{X\mathrm{Mean}_t(j_1), \ldots, X\mathrm{Mean}_t(j_s)\}$.

(viii) $\mathrm{Rel}_t := \mathrm{Rel}_t + \mathrm{score}_t(\mathrm{o}_s)$.

(ix) Next s.

(x) $\mathrm{Rel}_t := \mathrm{Rel}_t/k$// (the reliability index is calculated via formula (7)).

(xi) If $\mathrm{Rel}_t > \max\mathrm{Rel}_t$, then the linguistic label of the fuzzy set corresponding to the unknown with maximum mean solution is assigned to the output vector $O = [o_1, \ldots, o_k]$.

(xii) Next t with $t = 1, \ldots, T$.

(xiii) Next p with $p = 1, \ldots, P$.

At the end of the process, the user can create a thematic map of a specific output variable over the area of study and also a thematic map of the reliability index value obtained for the output variable. If the SFRE related to a specific subzone is inconsistent, the expert can decide whether or not eliminate rows to find solutions: in the first case, he decides that the symptoms associated to the rows that make the system inconsistent are not considered and eliminates them, so reducing the number of the equations. In the second case, he decides that the correspondent output variable for this subzone remains unknown and it is classified as unknown on the map.

4. Simulation Results

Here we show the results of an experiment in which we apply our method to census statistical data agglomerated on four districts of the east zone of Naples (Italy) (Figure 5). We use the year 2000 census data provided by the ISTAT (Istituto Nazionale di Statistica). These data contain information on population, buildings, housing, family, employment work for each census zone of Naples. Every district is considered as a subzone with homogeneous input data given in Table 4.

In this experiment, we consider the following four output variables: "o_1 = *Economic prosperity*" (wealth and prosperity of citizens), "o_2 = *Transition into the job*" (ease of finding work), "o_3 = *Social Environment*" (cultural levels of citizens), and "o_4 = *Housing development*" (presence of building and residential dwellings of new construction). For each variable,

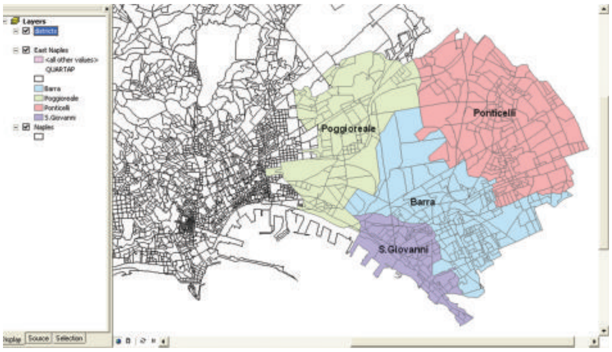

Figure 5: Area of study: four districts at east of Naples (Italy).

Table 3: Values of the TFNs low, mean, high.

Output	Low			Mean			High		
	INF	MEAN	SUP	INF	MEAN	SUP	INF	MEAN	SUP
o_1	0.0	0.3	0.5	0.3	0.5	0.8	0.5	0.8	1.0
o_2	0.0	0.3	0.5	0.3	0.5	0.8	0.5	0.8	1.0
o_3	0.0	0.3	0.5	0.3	0.5	0.8	0.5	0.8	1.0
o_4	0.0	0.3	0.5	0.3	0.5	0.8	0.5	0.8	1.0

Table 4: Input data obtained for the four subzones.

District	i_1	i_2	i_3	i_4	i_5	i_6	i_7
Barra	0.604	0.227	0.039	0.032	0.111	0.424	0.067
Poggioreale	0.664	0.297	0.060	0.051	0.086	0.338	0.149
Ponticelli	0.609	0.253	0.039	0.042	0.156	0.372	0.159
S. Giovanni	0.576	0.244	0.041	0.031	0.054	0.353	0.097

we create a fuzzy partition composed by three TFNs called "low", "mean", and "high" presented in Table 3.

Moreover, we consider the following seven input parameters: i_1 = percentage of people employed = number of people employed/total work force, i_2 = percentage of women employed = number of women employed/number of people employed, i_3 = percentage of entrepreneurs and professionals = number of entrepreneurs and professionals/number of people employed, i_4 = percentage of residents graduated = numbers of residents graduated/number of residents with age > 6 years, i_5 = percentage of new residential buildings = number of residential buildings built since 1982/total number of residential buildings, i_6 = percentage of residential dwellings owned = number of residential dwellings owned/total number of residential dwellings, and i_7 = percentage of residential dwellings with central heating system = number of residential dwellings with central heating system/total number of residential dwellings. In Table 4, we show these input data for the four subzones.

For the fuzzification process of the input data, the expert indicates a fuzzy partition for each input domain formed from three TFNs labeled "low", "mean", and "high", whose values are reported in Table 5. In Tables 6 and 7, we show the values obtained for the 21 symptoms $b_1, \ldots, b_{21}$; moreover, we report the input variable and the linguistic label of the correspondent TFN for each symptom b_i. In order to form the SFRE (1) in each subzone, the expert defines the equations by setting the impact values a_{ij} by basing over the most significant symptoms.

Now we illustrate this procedure for each subzone.

4.1. *Subzone "Barra"*

The expert chooses the significant symptoms b_2, b_4, b_5, b_7, b_{10}, b_{11}, b_{15}, b_{17}, b_{18}, b_{19}, by obtaining an SFRE (1) with $m = 10$ equations and $n = 12$ unknowns (Table 8).

The matrix A of the impact values a_{ij} has dimensions 10×12 and the vector B of the symptoms b_i has dimension 10×1 and both are given below. The SFRE (1) is inconsistent and eliminating the rows for which the value $\mathrm{IND}(j) = 0$, we obtain four maximal interval solutions $X_{\max(t)}$ $(t = 1, \ldots, 4)$ and we calculate the vector column $X\mathrm{Mean}_t$ on each maximal interval solution. Hence we associate to the output variable o_s $(s = 1, \ldots, 4)$, the linguistic label of the fuzzy set with the higher value calculated with formula (6) obtained for the corresponding unknowns $x_{j_1}, \ldots, x_{j_s}$ and given in Table 8. For determining the reliability of our solutions, we use the index given by formula (7). We obtain that $\mathrm{Rel}_t(o_1) = \mathrm{Rel}_t(o_2) = \mathrm{Rel}_t(o_3) = \mathrm{Rel}_t(o_4) = 0.6025$ for $t = 1, \ldots, 4$ and hence $\mathrm{Rel}(O) = \max\{\mathrm{Rel}_t(O) : t = 1, \ldots, 4\} = 0.6025$ where $O = \{o_1, \ldots o_4\}$. We note that the same final set of linguistic labels associated to the output variables o_1 = "high", o_2 = "mean", o_3 = "low", and o_4 = "low" is obtained as well. The relevant quantities are given below.

$$A = \begin{pmatrix} 0.5 & 1.0 & 0.0 & 0.4 & 1.0 & 0.2 & 0.2 & 0.7 & 0.3 & 0.1 & 0.3 & 0.2 \\ 0.3 & 0.5 & 0.2 & 0.4 & 0.5 & 0.4 & 0.3 & 0.6 & 0.2 & 0.0 & 0.0 & 0.0 \\ 0.2 & 0.7 & 0.2 & 0.2 & 0.7 & 0.2 & 0.2 & 0.7 & 0.2 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.2 & 0.0 & 0.8 & 0.3 & 0.1 & 0.8 & 0.2 & 0.2 & 0.3 & 0.0 & 0.0 \\ 0.5 & 0.3 & 0.1 & 0.6 & 0.4 & 0.1 & 0.6 & 0.4 & 0.1 & 0.1 & 0.0 & 0.0 \\ 0.3 & 0.7 & 0.3 & 0.3 & 0.7 & 0.3 & 0.2 & 0.7 & 0.3 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.2 & 0.1 & 0.2 & 0.1 & 0.1 & 0.1 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.2 & 0.1 & 0.4 & 0.1 & 0.2 & 0.5 & 0.1 & 0.3 & 0.7 & 0.3 \\ 0.1 & 0.4 & 0.4 & 0.1 & 0.4 & 0.4 & 0.1 & 0.5 & 0.5 & 0.2 & 0.4 & 0.5 \\ 0.5 & 0.2 & 0.0 & 0.4 & 0.3 & 0.0 & 0.4 & 0.3 & 0.0 & 1.0 & 0.1 & 0.0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.98 \\ 0.36 \\ 0.63 \\ 1.00 \\ 0.40 \\ 0.60 \\ 0.10 \\ 0.59 \\ 0.41 \\ 1.00 \end{pmatrix}, \tag{33}$$

Table 5: TFNs values for the input domains.

Input variable	Low			Mean			High		
	INF	MEAN	SUP	INF	MEAN	SUP	INF	MEAN	SUP
i_1	0.00	0.40	0.60	0.40	0.60	0.80	0.60	0.80	1.00
i_2	0.00	0.10	0.30	0.10	0.30	0.40	0.30	0.50	1.00
i_3	0.00	0.04	0.06	0.04	0.06	0.10	0.07	0.20	1.00
i_4	0.00	0.02	0.04	0.02	0.04	0.07	0.04	0.07	1.00
i_5	0.00	0.05	0.08	0.05	0.08	0.10	0.08	0.10	1.00
i_6	0.00	0.10	0.30	0.10	0.30	0.60	0.30	0.60	1.00
i_7	0.00	0.10	0.30	0.10	0.30	0.50	0.30	0.50	1.00

Table 6: TFNs for the symptoms $b_1 \div b_{12}$.

Subzone	b_1: i_1 = low	b_2: i_1= mean	b_3: i_1 = high	b_4: i_2 = low	b_5: i_2 = mean	b_6: i_2 = high	b_7: i_3 = low	b_8: i_3 = mean	b_9: i_3 = high	b_{10}: i_4 = low	b_{11}: i_4 = mean	b_{12}: i_4 = high
Barra	0.00	0.98	0.02	0.36	0.63	0.00	1.00	0.00	0.00	0.40	0.60	0.00
Poggioreale	0.00	0.93	0.07	0.01	0.99	0.00	0.00	1.00	0.00	0.00	0.63	0.37
Ponticelli	0.00	0.91	0.05	0.23	0.76	0.00	1.00	0.00	0.00	0.00	0.93	0.07
S. Giovanni	0.12	0.88	0.00	0.28	0.72	0.00	0.95	0.05	0.00	0.45	0.55	0.00

$$X_{\max(1)} = \begin{pmatrix} [0.40, 0.40] \\ [0.36, 0.36] \\ [0.00, 1.00] \\ [0.00, 0.36] \\ [0.00, 1.00] \\ [0.00, 0.36] \\ [0.00, 1.00] \\ [0.00, 0.36] \\ [0.41, 0.41] \\ [1.00, 1.00] \\ [0.00, 0.10] \\ [0.00, 0.10] \end{pmatrix}, \quad X_{\max(2)} = \begin{pmatrix} [0.40, 0.40] \\ [0.00, 0.36] \\ [0.00, 1.00] \\ [0.36, 0.36] \\ [0.00, 1.00] \\ [0.00, 0.36] \\ [0.00, 1.00] \\ [0.00, 0.36] \\ [0.41, 0.41] \\ [1.00, 1.00] \\ [0.00, 0.10] \\ [0.00, 0.10] \end{pmatrix},$$

$$X_{\max(3)} = \begin{pmatrix} [0.40, 0.40] \\ [0.00, 0.36] \\ [0.00, 1.00] \\ [0.00, 0.36] \\ [0.00, 1.00] \\ [0.36, 0.36] \\ [0.00, 1.00] \\ [0.00, 0.36] \\ [0.41, 0.41] \\ [1.00, 1.00] \\ [0.00, 0.10] \\ [0.00, 0.10] \end{pmatrix}, \quad X_{\max(4)} = \begin{pmatrix} [0.40, 0.40] \\ [0.00, 0.36] \\ [0.00, 1.00] \\ [0.36, 0.36] \\ [0.00, 1.00] \\ [0.00, 0.36] \\ [0.00, 1.00] \\ [0.36, 0.36] \\ [0.41, 0.41] \\ [1.00, 1.00] \\ [0.00, 0.10] \\ [0.00, 0.10] \end{pmatrix},$$

$$XMean_1 = \begin{pmatrix} 0.40 \\ 0.36 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.18 \\ 0.41 \\ 1.00 \\ 0.05 \\ 0.05 \end{pmatrix}, \quad XMean_2 = \begin{pmatrix} 0.40 \\ 0.18 \\ 0.50 \\ 0.36 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.18 \\ 0.41 \\ 1.00 \\ 0.05 \\ 0.05 \end{pmatrix},$$

$$XMean_3 = \begin{pmatrix} 0.40 \\ 0.18 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.36 \\ 0.50 \\ 0.18 \\ 0.18 \\ 1.00 \\ 0.05 \\ 0.05 \end{pmatrix}, \quad XMean_4 = \begin{pmatrix} 0.40 \\ 0.18 \\ 0.05 \\ 0.36 \\ 0.50 \\ 0.18 \\ 0.50 \\ 0.36 \\ 0.41 \\ 1.00 \\ 0.05 \\ 0.05 \end{pmatrix}. \tag{34}$$

4.2. Subzone "Poggioreale". The expert chooses the significant symptoms b_2, b_5, b_8, b_{11}, b_{12}, b_{14}, b_{15}, b_{17}, b_{18}, b_{19}, b_{20}, by obtaining an SFRE (1) with $m = 11$ equations and $n = 12$ unknowns (Table 9). The matrix A of the impact values a_{ij} has dimension 11×12 and the vector B of the symptoms b_i has dimension 11×1 which are given below. The SFRE (1)

Table 7: TFNs for the symptoms $b_{13} \div b_{21}$.

Subzone	b_{13}: i_5 = low	b_{14}: i_5 = mean	b_{15}: i_5 =high	b_{16}: i_6 = low	b_{17}: i_6 = mean	b_{18}: i_6 = high	b_{19}: i_7 = low	b_{20}: i_7 = mean	b_{21}: i_7 = high
Barra	0.00	0.00	0.10	0.00	0.59	0.41	1.00	0.00	0.00
Poggioreale	0.00	0.70	0.30	0.00	0.87	0.13	0.75	0.25	0.00
Ponticelli	0.00	0.00	1.00	0.00	0.76	0.24	0.70	0.30	0.00
S. Giovanni	0.87	0.13	0.00	0.00	0.82	0.18	1.00	0.00	0.00

Table 8: Final linguistic labels for the output variables in the district Barra.

Output variable	$\mathrm{Score}_1(o_s)$	$\mathrm{Score}_2(o_s)$	$\mathrm{Score}_3(o_s)$	$\mathrm{Score}_4(o_s)$
o_1	high	high	high	high
o_2	mean	mean	mean	mean
o_3	low	low	low	low
o_4	low	low	low	low

is inconsistent and eliminating the rows for which the value $\mathrm{IND}(j) = 0$, we obtain 12 maximal interval solutions $X_{\max(t)}$ ($t = 1, \dots, 12$), and we calculate the vector column $X\mathrm{Mean}_t$ on each maximal interval solution. The relevant quantities are given below

$$A = \begin{pmatrix} 0.5 & 1.0 & 0.0 & 0.4 & 1.0 & 0.2 & 0.2 & 0.7 & 0.3 & 0.1 & 0.3 & 0.2 \\ 0.2 & 1.0 & 0.2 & 0.2 & 1.0 & 0.2 & 0.2 & 0.9 & 0.2 & 0.0 & 0.0 & 0.0 \\ 0.2 & 1.0 & 0.2 & 0.2 & 1.0 & 0.2 & 0.2 & 1.0 & 0.2 & 0.0 & 0.0 & 0.0 \\ 0.3 & 0.7 & 0.3 & 0.3 & 0.7 & 0.3 & 0.2 & 0.7 & 0.3 & 0.1 & 0.2 & 0.2 \\ 0.4 & 0.5 & 0.6 & 0.3 & 0.5 & 0.6 & 0.3 & 0.5 & 0.6 & 0.0 & 0.0 & 0.1 \\ 0.3 & 0.7 & 0.3 & 0.3 & 0.7 & 0.3 & 0.2 & 0.7 & 0.3 & 0.1 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.6 & 0.3 & 0.4 & 0.6 & 0.2 & 0.4 & 0.6 & 0.0 & 0.1 & 0.2 \\ 0.1 & 0.9 & 0.1 & 0.1 & 0.9 & 0.1 & 0.2 & 0.8 & 0.2 & 0.2 & 0.8 & 0.2 \\ 0.0 & 0.1 & 0.5 & 0.1 & 0.2 & 0.5 & 0.1 & 0.2 & 0.5 & 0.0 & 0.1 & 0.4 \\ 0.4 & 0.1 & 0.0 & 0.8 & 0.5 & 0.3 & 0.5 & 0.3 & 0.1 & 0.7 & 0.3 & 0.0 \\ 0.1 & 0.2 & 0.1 & 0.1 & 0.2 & 0.1 & 0.1 & 0.2 & 0.1 & 0.3 & 0.6 & 0.2 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.93 \\ 0.99 \\ 1.0 \\ 0.63 \\ 0.37 \\ 0.7 \\ 0.3 \\ 0.87 \\ 0.13 \\ 0.75 \\ 0.25 \end{pmatrix}, \tag{35}$$

$$X_{\max(1)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.13, 0.13] \\ [0.75, 0.75] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.25, 0.25] \\ [0.0, 0.25] \\ [0.0, 0.13] \end{pmatrix}, \quad X_{\max(2)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.13, 0.13] \\ [0.75, 0.75] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 0.25] \\ [0.25, 0.25] \\ [0.0, 0.13] \end{pmatrix},$$

$$X_{\max(3)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.13, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.25, 0.25] \\ [0.0, 0.25] \\ [0.0, 0.13] \end{pmatrix}, \quad X_{\max(4)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.13, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 0.25] \\ [0.25, 0.25] \\ [0.0, 0.13] \end{pmatrix},$$

$$X_{\max(5)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.13, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.25, 0.25] \\ [0.0, 0.25] \\ [0.0, 0.13] \end{pmatrix}, \quad X_{\max(6)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.13, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 0.25] \\ [0.25, 0.25] \\ [0.0, 0.13] \end{pmatrix},$$

Table 9: Final linguistic labels for the output variables in the district Poggioreale.

Output	Score_1 (o_s)	Score_2 (o_s)	Score_3 (o_s)	Score_4 (o_s)	Score_5 (o_s)	Score_6 (o_s)	Score_7 (o_s)	Score_8 (o_s)	Score_9 (o_s)	Score_{10} (o_s)	Score_{11} (o_s)	Score_{12} (o_s)
o_1	low	low	low	high	low	low	low	high	low	low	low	high
o_2	low	low	low	mean	low	low	low	mean	low	low	low	mean
o_3	low	low	low	low	low	low	low	low	low	low	low	low
o_4	low	mean	low	mean	low	mean	low	mean	low	mean	low	mean

$$X_{\max(7)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.13, 0.13] \\ [0.0, 0.13] \\ [0.25, 0.25] \\ [0.0, 0.25] \\ [0.0, 0.13] \end{pmatrix}, \qquad X_{\max(8)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.13, 0.13] \\ [0.0, 0.13] \\ [0.0, 0.25] \\ [0.25, 0.25] \\ [0.0, 0.13] \end{pmatrix}, \qquad X\text{Mean}_1 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.13 \\ 0.75 \\ 0.065 \\ 0.065 \\ 0.5 \\ 0.065 \\ 0.065 \\ 0.25 \\ 0.125 \\ 0.05 \end{pmatrix}, \qquad X\text{Mean}_2 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.13 \\ 0.75 \\ 0.065 \\ 0.065 \\ 0.5 \\ 0.065 \\ 0.065 \\ 0.125 \\ 0.25 \\ 0.065 \end{pmatrix},$$

$$X_{\max(9)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.13, 0.13] \\ [0.25, 0.25] \\ [0.0, 0.25] \\ [0.0, 0.13] \end{pmatrix}, \qquad X_{\max(10)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.13, 0.13] \\ [0.0, 0.25] \\ [0.25, 0.25] \\ [0.0, 0.13] \end{pmatrix}, \qquad X\text{Mean}_3 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.13 \\ 0.065 \\ 0.5 \\ 0.065 \\ 0.065 \\ 0.25 \\ 0.125 \\ 0.065 \end{pmatrix}, \qquad X\text{Mean}_4 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.13 \\ 0.065 \\ 0.5 \\ 0.065 \\ 0.065 \\ 0.125 \\ 0.25 \\ 0.065 \end{pmatrix},$$

$$X_{\max(11)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.25, 0.25] \\ [0.0, 0.25] \\ [0.13, 0.13] \end{pmatrix}, \qquad X_{\max(12)} = \begin{pmatrix} [0.37, 0.37] \\ [0.0, 0.3] \\ [0.0, 0.13] \\ [0.75, 0.75] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 1.0] \\ [0.0, 0.13] \\ [0.0, 0.13] \\ [0.0, 0.25] \\ [0.25, 0.25] \\ [0.13, 0.13] \end{pmatrix}, \qquad X\text{Mean}_5 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.065 \\ 0.13 \\ 0.5 \\ 0.065 \\ 0.065 \\ 0.25 \\ 0.125 \\ 0.05 \end{pmatrix}, \qquad X\text{Mean}_6 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.065 \\ 0.13 \\ 0.5 \\ 0.065 \\ 0.065 \\ 0.125 \\ 0.25 \\ 0.05 \end{pmatrix},$$

$$
XMean_7 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.065 \\ 0.065 \\ 0.5 \\ 0.13 \\ 0.065 \\ 0.25 \\ 0.125 \\ 0.065 \end{pmatrix}, \quad XMean_8 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.065 \\ 0.065 \\ 0.5 \\ 0.13 \\ 0.065 \\ 0.125 \\ 0.25 \\ 0.065 \end{pmatrix},
$$
$$
XMean_9 = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.065 \\ 0.065 \\ 0.5 \\ 0.065 \\ 0.13 \\ 0.25 \\ 0.125 \\ 0.05 \end{pmatrix}, \quad XMean_{10} = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.065 \\ 0.065 \\ 0.5 \\ 0.065 \\ 0.13 \\ 0.125 \\ 0.25 \\ 0.05 \end{pmatrix},
$$
$$
XMean_{11} = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.065 \\ 0.065 \\ 0.5 \\ 0.065 \\ 0.065 \\ 0.25 \\ 0.125 \\ 0.13 \end{pmatrix}, \quad XMean_{12} = \begin{pmatrix} 0.37 \\ 0.15 \\ 0.065 \\ 0.75 \\ 0.065 \\ 0.065 \\ 0.5 \\ 0.065 \\ 0.065 \\ 0.125 \\ 0.25 \\ 0.13 \end{pmatrix}. \tag{36}
$$

For determining the reliability of our solutions, we use the index given by formula (7). We obtain $\mathrm{Rel}(O_k) = 0.4675$ for $k = 1, \ldots, 12$. Then we obtain two final sets of linguistic labels associated to the output variables: o_1 = "low", o_2 = "low", o_3 = "low", o_4 = "low", and o_1 = "low", o_2 = "low", o_3 = "low", o_4 = "mean", with a same reliability index value 0.4675. The expert prefers to choose the second solution: o_1 = "low", o_2 = "low", o_3 = "low", o_4 = "mean" because he considers that in the last two years in this district the presence of building and residential dwellings of new construction has increased although marginally. We obtain four final thematic maps shown in Figures 6, 7, 8, 9 for the output variable o_1, o_2, o_3, o_4, respectively.

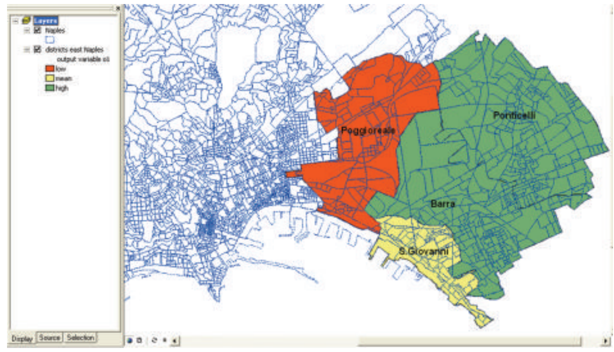

FIGURE 6: Thematic map for output variable o_1 (*Economic prosperity*).

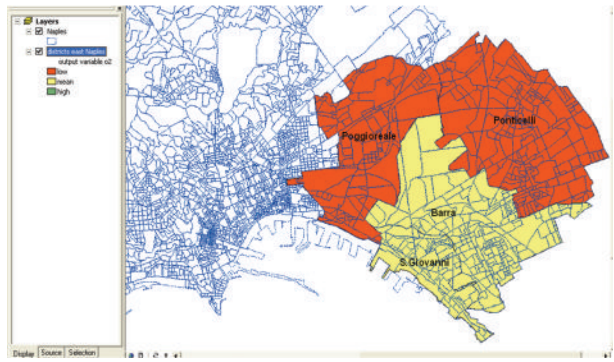

FIGURE 7: Thematic map of the output variable o_2 (*Transition into the job*).

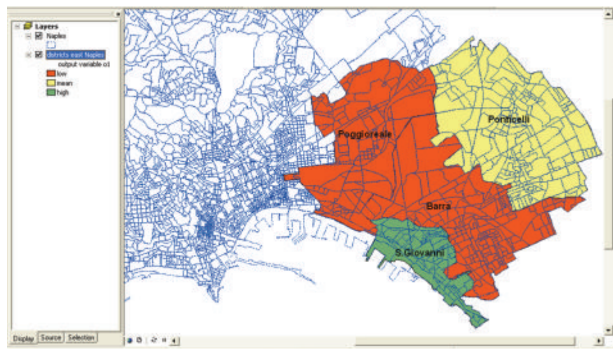

FIGURE 8: Thematic map for the output variable o_3 (*Social Environment*).

The results show that there was no housing development in the four districts in the last 10 years, and there is difficulty

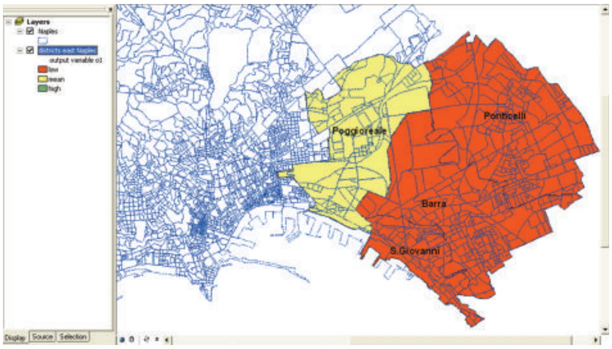

Figure 9: Thematic map for the output variable o_4 (*Housing development*).

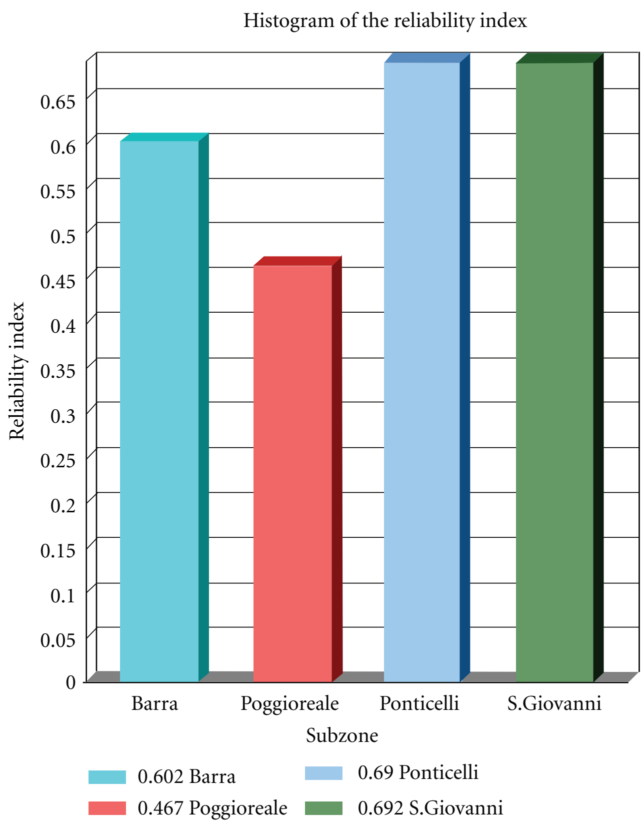

Figure 10: Histogram of the reliability index Rel(O) for the four subzones.

in finding job positions. In Figure 10, we show the histogram of the reliability index Rel(O) for each subzone, where $O = [o_1, o_2, o_3, o_4]$.

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