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Research Article

Some Classes of Difference Sequence Spaces of Fuzzy Real Numbers Defined by Orlicz Function

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We introduce the classes of generalized difference bounded, convergent, and null sequences of fuzzy real numbers defined by an Orlicz function. Some properties of these sequence spaces like solidness, symmetricity, and convergence-free are studied. We obtain some inclusion relations involving these sequence spaces.

1. Introduction

The concept of fuzzy set theory was introduced by Zadeh in 1965. Later on sequences of fuzzy numbers have been discussed by Syau [1], Tripathy and Baruah [2], Tripathy and Borgohain [3], Tripathy and Dutta [4, 5], Tripathy and Sarma [6, 7], and many others.

Kizmaz [8] defined the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$ of complex numbers as follows:

$$Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\}, \quad \text{for } Z = \ell_\infty, c, c_0, \\ \text{where } \Delta x = (\Delta x_k) = (x_k - x_{k+1}). \tag{1}$$

The above spaces are Banach spaces, normed by

$$\|x\|_\Delta = |x_1| + \sup_k |\Delta x_k|. \tag{2}$$

The idea of Kizmaz [8] was applied to introduce different type of difference sequence spaces and study their different properties by Et et al. [9], Tripathy et al. [10], Tripathy and Baruah [2], Tripathy and Borgohain [3], Tripathy and Esi [11], Tripathy et al. [12], Tripathy and Mahanta [13], and many others.

Tripathy and Esi [11] introduced a new type of difference sequence spaces as follows. Let $m \in N$ be fixed, then

$$Z(\Delta_m) = \{x = (x_k) : (\Delta_m x_k) \in Z\}, \quad \text{for } Z = \ell_\infty, c, c_0, \\ \text{where } \Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m}). \tag{3}$$

The above sequence spaces are Banach spaces, normed by

$$\|x\|_\Delta = \sum_{r=1}^{m} |x_r| + \sup_k |\Delta_m x_k|. \tag{4}$$

Tripathy et al. [12] further generalized this notion and introduced the following. For $m \geq 1$ and $n \geq 1$,

$$Z(\Delta_m^n) = \{x = (x_k) : (\Delta_m^n x_k) \in Z\}, \quad \text{for } Z = \ell_\infty,\ c,\ c_0, \\ \text{where } \Delta_m^n x_k = \Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m}, \quad \forall k \in N. \tag{5}$$

This generalized difference has the following binomial representation:

$$\Delta_m^n x_k = \sum_{r=0}^{n} (-1)^r \binom{n}{r} x_{k+rm}. \tag{6}$$

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$, which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$ (one may refer to Korasnoselkii and Rutitsky [14]).

An Orlicz function M is said to satisfy Δ_2-condition for all values of x, if there exists a constant $K > 0$, such that $M(Lx) \leq KLM(x)$, for all $x > 0$ and for $L > 1$.

Remark 1. An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$, for all λ with $0 < \lambda < 1$.

Throughout the paper $w^F, \ell^F, \ell^F_\infty$ represent the classes of *all*, *absolutely summable,* and *bounded* sequences of fuzzy real numbers, respectively.

2. Definitions and Background

Let $C(R^n) = \{A \subset R^n : A \text{ is compact and convex}\}$. Then the space $C(R^n)$ has linear structure induced by the operations $A + B = \{a + b : a \in A,\ b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$ for $A, B \in C(R^n)$ and $\lambda \in R$.

The Hausdorff distance between A and B of $C(R^n)$ is defined as

$$\delta_\infty(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}. \quad (7)$$

It is well known that $(C(R^n), \delta_\infty)$ is a complete metric space. A fuzzy real number on R^n is a function $X : R^n \to I(= [0,1])$ associating each real number $t \in R^n$ with its grade of membership $X(t)$.

A fuzzy real number X is called *convex* if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

If there exists $t_0 \in R^n$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*.

A fuzzy real number X is said to be *upper semicontinuous* if for each $\varepsilon > 0$, $X^{-1}([0, a+\varepsilon))$, for all $a \in I$ is open in the usual topology of R^n.

The class of all *upper semi-continuous*, *normal*, *convex* fuzzy real numbers is denoted by $R^n(I)$. Let $X \in R^n(I)$, then the α-level set X^α, for $0 < \alpha \leq 1$, is defined by, $X^\alpha = \{t \in R^n : X(t) \geq \alpha\}$ and is a nonempty compact convex subset of R^n. The 0-level set, that is, X^0, is the closure of strong 0-cut, that is, $X^0 = cl\{t \in R^n : X(t) > 0\}$. The absolute value of $X \in R(I)$, that is, $|X|$, is defined by

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{for } t \geq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

For $r \in R^n$, $\overline{r} \in R^n(I)$ is defined as,

$$\overline{r}(t) = \begin{cases} 1, & \text{for } t = r, \\ 0, & \text{otherwise.} \end{cases} \quad (9)$$

The additive identity and multiplicative identity of $R^n(I)$ are denoted by $\overline{\theta}$ and $\overline{e}$, respectively, where $\theta = (0,0,\ldots,0)$ and $e = (1,1,\ldots,1)$. The zero sequence of fuzzy real numbers is denoted by $\overline{\vartheta} = \{\overline{\theta}, \overline{\theta}, \ldots, \overline{\theta}, \ldots\}$.

The linear structure of $C(R^n)$ induces the addition $X + Y$ and scalar multiplication λX, $\lambda \in R$ in terms of α-level sets, by $[X+Y]^\alpha = [X]^\alpha + [Y]^\alpha$ and $[\lambda X]^\alpha = \lambda[X]^\alpha$, for each $0 < \alpha \leq 1$, where

$$\lambda X(t) = \begin{cases} \overline{0}, & \text{for } \lambda = 0, \\ X(\lambda^{-1}t), & \text{otherwise.} \end{cases} \quad (10)$$

Define, for each $1 \leq q < \infty$,

$$d_q(X,Y) = \left(\int_0^1 \delta_\infty(X^\alpha, Y^\alpha)^q d\alpha\right)^{1/q} \quad (11)$$

and $d_\infty : R^n(I) \times R^n(I) \to R$ such that $d_\infty(X,Y) = \sup_{0<\alpha\leq 1}\delta_\infty(X^\alpha, Y^\alpha)$, where δ_∞ is the Hausdorff metric. Clearly, $d_\infty(X,Y) = \lim_{q\to\infty} d_q(X,Y)$ with $d_q \leq d_r$ if $q \leq r$. Moreover $(R^n(I), d_\infty)$ is a complete, separable, and locally compact metric space.

A sequence $X = (X_k)$ of fuzzy real numbers is said to converge to the fuzzy number X_0, if for every $\varepsilon > 0$, there exists $k_0 \in N$ such that $d_\infty(X_k, X_0) < \varepsilon$, for all $k \geq k_0$.

A sequence space E is said to be *solid* if $(Y_n) \in E$, whenever $(X_n) \in E$ and $|Y_n| \leq |X_n|$, for all $n \in N$.

Let $X = (X_n)$ be a sequence, then $S(X)$ denotes the set of all permutations of the elements of (X_n), that is, $S(X) = \{(X_{\pi(n)}) : \pi \text{ is a permutation of } N\}$. A sequence space E is said to be *symmetric* if $S(X) \subset E$ for all $X \in E$.

A sequence space E is said to be *convergence-free* if $(Y_n) \in E$ whenever $(X_n) \in E$ and $X_n = \overline{\theta}$ implies $Y_n = \overline{\theta}$.

A sequence space E is said to be *monotone* if E contains the canonical preimages of all its step spaces.

Lemma 2. *A class of sequences E is solid which implies that E is monotone.*

Lindenstrauss and Tzafriri [15] used the notion of Orlicz function and introduced the sequence space:

$$\ell_M = \left\{(x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ \text{for some } \rho > 0\right\}. \quad (12)$$

The space ℓ_M with the norm,

$$\|(x_k)\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}, \quad (13)$$

becomes a Banach space, which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p, which is an Orlicz sequence space with $M(x) = x^p$, for $1 \leq p < \infty$.

In the later stage different classes of Orlicz sequence spaces were introduced and investigated by Altin et al. [16], Et et al. [9], Tripathy et al. [10], Tripathy and Borgohain [3], Tripathy and Hazarika [17], Tripathy and Mahanta [13], Tripathy and Sarma [6, 7, 18], and many others.

In this paper we introduce the following difference sequence spaces:

$$\ell_\infty^F(M,\Delta_m^n) = \left\{(X_k) \in w^F : \sup_k \left(\frac{d_\infty\left(\Delta_m^n X_k, \overline{\theta}\right)}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}$$

$$c^F(M,\Delta_m^n) = \left\{(X_k) \in w^F : \lim_{k\to\infty} \left(\frac{d_\infty\left(\Delta_m^n X_k, L\right)}{\rho}\right) = 0, \text{ for some } \rho > 0, L \in R^n(I)\right\}$$

$$c_0^F(M,\Delta_m^n) = \left\{(X_k) \in w^F : \lim_{k\to\infty} \left(\frac{d_\infty\left(\Delta_m^n X_k, \overline{\theta}\right)}{\rho}\right) = 0, \text{ for some } \rho > 0\right\}. \tag{14}$$

3. Main Results

Theorem 3. *The classes of sequences $\ell_\infty^F(M,\Delta_m^n)$, $c^F(M,\Delta_m^n)$, $c_0^F(M,\Delta_m^n)$ are complete metric spaces by the metric*

$$\eta(X,Y) = \sum_{r=1}^{mn} d_\infty(X_r,Y_r) + \inf\left\{\rho > 0 : \sup_k M\left(\frac{d_\infty\left(\Delta_m^n X_k, \Delta_m^n Y_k\right)}{\rho}\right) \le 1\right\}, \tag{15}$$

for $X, Y \in \ell_\infty(M,\Delta_m^n)^F$, $c^F(M,\Delta_m^n)$, $c_0^F(M,\Delta_m^n)$.

Proof. We establish the result for the class of sequences $\ell_\infty^F(M,\Delta_m^n)$. The proof for the other cases will follow similarly. It can easily be verified that $\ell_\infty^F(M,\Delta_m^n)$ is a metric space by the metric η defined above. Next we show that it is a complete metric space.

Let $(X^{(i)})$ be a Cauchy sequence in $\ell_\infty^F(M,\Delta_m^n)$ such that $X^{(i)} = (X_n^{(i)})_{n=1}^{\infty}$. Let $\varepsilon > 0$ be given. For a fixed $x_0 > 0$, choose $r > 0$ such that $M(rx_0/2) \ge 1$. Then there exits a positive integer $n_0 = n_0(\varepsilon)$ such that

$$\eta\left(X^{(i)},X^{(j)}\right) < \frac{\varepsilon}{rx_0}, \quad \forall i,j \ge n_0. \tag{16}$$

By the definition of η, we have,

$$\sum_{r=1}^{mn} d_\infty\left({X_r}^{(i)},{X_r}^{(j)}\right) + \inf\left\{\rho > 0 : \sup_k M\left(\frac{d_\infty\left(\Delta_m^n X_r^{(i)}, \Delta_m^n X_r^{(j)}\right)}{\rho}\right) \le 1\right\} \le \varepsilon, \quad \forall i,j \ge n_0, \tag{17}$$

which implies

$$\sum_{r=1}^{mn} d_\infty\left(X_r^{(i)},X_r^{(j)}\right) < \varepsilon, \quad \forall i,j \ge n_0,$$
$$\Longrightarrow d_\infty\left(X_r^{(i)},X_r^{(j)}\right) < \varepsilon, \quad \forall i,j \ge n_0,\ r = 1,2,3,\dots,mn. \tag{18}$$

Hence $(X_r^{(i)})$, for $r = 1,2,3,\dots,mn$ are Cauchy sequence in $R^n(I)$ and hence are convergent in $R^n(I)$, since $R^n(I)$ is a complete metric space.

Let

$$\lim_{i\to\infty} X_r^{(i)} = X_r, \quad \text{for } r = 1,2,3,\dots,mn. \tag{19}$$

Also,

$$\sup_k M\left(\frac{d_\infty\left(\Delta_m^n X_k^{(i)}, \Delta_m^n X_k^{(j)}\right)}{\rho}\right) \le 1, \quad \forall i,j \ge n_0,$$
$$\Longrightarrow M\left(\frac{d_\infty\left(\Delta_m^n X_k^{(i)}, \Delta_m^n X_k^{(j)}\right)}{\eta(X^{(i)},X^{(j)})}\right) \le 1 \le M\left(\frac{rx_0}{2}\right), \quad \forall i,j \ge n_0. \tag{20}$$

Since M is continuous, we get,

$$d_\infty\left(\Delta_m^n X_k^{(i)}, \Delta_m^n X_k^{(j)}\right) \le \frac{rx_0}{2}\cdot\eta\left(X^{(i)},X^{(j)}\right), \quad \forall i,j \ge n_0,$$
$$\Longrightarrow d_\infty\left(\Delta_m^n X_k^{(i)}, \Delta_m^n X_k^{(j)}\right) < \frac{rx_0}{2}\cdot\frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2}, \quad \forall i,j \ge n_0,$$
$$\Longrightarrow d_\infty\left(\Delta_m^n X_k^{(i)}, \Delta_m^n X_k^{(j)}\right) < \frac{\varepsilon}{2}, \quad \forall i,j \ge n_0, \tag{21}$$

which implies $(\Delta_m^n X_k^{(i)})$ is a Cauchy sequence in $R^n(I)$ and so is convergent in $R^n(I)$, since $R^n(I)$ is complete metric space.

Let $\lim_i \Delta_m^n X_k^{(i)} = Y_k$ (say), in $R^n(I)$, for each $k \in N$.

We have to prove that

$$\lim_i X^{(i)} = X, \quad X \in \ell_\infty^F(M,\Delta_m^n). \tag{22}$$

For $k = 1$, we have, from (6) and (19),

$$\lim_{i\to\infty} X_{mn+1}^{(i)} = X_{mn+1}, \quad \text{for } m \ge 1,\ n \ge 1. \tag{23}$$

Proceeding in this way inductively, we get

$$\lim_{i\to\infty} X_k^{(i)} = X_k, \quad \text{for each } k \in N. \tag{24}$$

Also, $\lim_i \Delta_m^n X_k^{(i)} = \Delta_m^n X_k$, for each $k \in N$.

Next taking $j \to \infty$, keeping i fixed, and by the continuity of M, we have the following from (20):

$$\sup_k M\left(\frac{d_\infty\left(\Delta_m^n X_k^{(i)}, \Delta_m^n X_k\right)}{\rho}\right) \le 1, \quad \text{for some } \rho > 0. \tag{25}$$

Now on taking the infimum of such ρ's, we get

$$\inf\left\{\rho > 0 : \sup_k M\left(\frac{d_\infty\left(\Delta_m^n X_k^{(i)}, \Delta_m^n X_k\right)}{\rho}\right) \le 1\right\} < \varepsilon, \quad \forall i \ge n_0 \ (\text{by}(2)). \tag{26}$$

Hence from (17) on taking limit as $j \to \infty$, we get

$$\sum_{r=1}^{mn} d_\infty\left(X_r^{(i)}, X_r\right) + \inf\left\{\rho > 0 : \sup_k M\left(\frac{d_\infty\left(\Delta_m^n X_k^{(i)}, \Delta_m^n X_k\right)}{\rho}\right) \le 1\right\} < \varepsilon + \varepsilon = 2\varepsilon, \quad \forall i \ge n_0, \tag{27}$$

which implies

$$\eta\left(X^{(i)},\ X\right) < 2\varepsilon, \quad \forall i \ge n_0. \tag{28}$$

That is, $\lim_i X^{(i)} = X$.

Next we show that $X \in \ell_\infty^F(M, \Delta_m^n)$.

We know that

$$d_\infty\left(\Delta_m^n X_k, \overline{\theta}\right) \le d_\infty\left(\Delta_m^n X_k^{(i)}, \Delta_m^n X_k\right) + d_\infty\left(\Delta_m^n X_k^{(i)}, \overline{\theta}\right). \tag{29}$$

Since M is continuous and nondecreasing, so we get

$$\sup_k M\left(\frac{d_\infty\left(\Delta_m^n X_k, \overline{\theta}\right)}{\rho}\right) \le \sup_k M\left(\frac{d_\infty\left(\Delta_m^n X_k^{(i)}, \Delta_m^n X_k\right)}{\rho}\right) + \sup_k M\left(\frac{d_\infty\left(\Delta_m^n X_k^{(i)}, \overline{\theta}\right)}{\rho}\right) < \infty, \tag{30}$$

which implies $X \in \ell_\infty^F(M, \Delta_m^n)$.

Hence $\ell_\infty^F(M, \Delta_m^n)$ is a complete metric space.

The other cases can be established similarly.

This completes the proof of the theorem. □

Result 1. The classes of sequences $\ell_\infty^F(M, \triangle_m^n)$, $c^F(M, \triangle_m^n)$, $c_0^F(M, \triangle_m^n)$, are neither solid nor monotone in general.

Proof. The result follows from the following example.

Example 4. Consider the sequence space $\ell_\infty^F(M, \Delta_m^n)$. Let $m = 2$ and $n = 3$. Let $M(x) = |x|$, for all $x \in [0, \infty)$.

Consider the sequence (X_k) defined by

$$X_k(t) = \begin{cases} 1, & \text{for } k \in N,\ t = (k, k, k, \ldots), \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

Then,

$$\Delta_2^3 X_k(t) = \begin{cases} 1, & \text{for } k \in N,\ t = (0, 0, 0, \ldots), \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$

Then, we have $d_\infty(\Delta_2^3 X_k, \overline{\theta}) = 0$, for all $k \in N$.

Hence, we have

$$\sup_k M\left(\frac{d_\infty\left(\Delta_2^3 X_k, \overline{\theta}\right)}{\rho}\right) < \infty, \quad \text{for some } \rho > 0, \tag{33}$$

which implies $(X_k) \in \ell_\infty^F(M, \Delta_2^3)$.

Consider the sequence (α_k) of scalars defined by

$$\alpha_k = \begin{cases} 1, & \text{for } k = i^2, i \in N, \\ 0, & \text{otherwise.} \end{cases} \tag{34}$$

For $k = i^2$, we have

$$\alpha_k X_k(t) = \begin{cases} 1, & t = (k, k, k, \ldots), \\ 0, & \text{otherwise.} \end{cases} \tag{35}$$

For $k \ne i^2$, we have

$$\alpha_k X_k(t) = \begin{cases} 1, & t = (0, 0, 0, \ldots), \\ 0, & \text{otherwise,} \end{cases} \tag{36}$$

which implies

$$\sup_k M\left(\frac{d_\infty\left(\Delta_2^3 X_k, \overline{\theta}\right)}{\rho}\right) = \infty, \quad \text{for each } \rho > 0. \tag{37}$$

Hence $(\alpha_k X_k) \notin \ell_\infty^F(M, \Delta_2^3)$.

Thus, $\ell_\infty^F(M, \triangle_m^n)$ is not solid in general. □

Similarly the other cases can be established. The classes of sequences are not monotone followed by Lemma 2.

Result 2. The classes of sequences $\ell_\infty^F(M, \triangle_m^n)$, $c^F(M, \triangle_m^n)$, and $c_0^F(M, \triangle_m^n)$ are not symmetric in general.

Proof. The result follows from the following example.

Example 5. Let $m = 2$ and $n = 1$. Let $M(x) = x^2$, for all $x \in [0, \infty)$. Consider the sequence (X_k) defined by

$$X_k(t) = \begin{cases} 1, & \text{for } k \in N,\ t = (k, k, k, \ldots), \\ 0, & \text{otherwise.} \end{cases} \tag{38}$$

Then,

$$\Delta_2 X_k(t) = \begin{cases} 1, & \text{for } k \in N,\ t = (-2,-2,-2,\ldots), \\ 0, & \text{otherwise.} \end{cases} \quad (39)$$

Then, $d_\infty(\Delta_2 X_k, \overline{0}) = 1$, for all $k \in N$, which shows $(X_k) \in c^F(M,\Delta_2) \subset \ell_\infty^F(M,\Delta_2)$.

Let (Y_k) be a rearrangement of (X_k) such that $(Y_k) = (X_1, X_2, X_4, X_3, X_9, X_5, X_{16}, X_6, X_{25}, \ldots)$. Then we get $d_\infty(\Delta_2 Y_k, \overline{0}) \approx k-(k-1)^2 \approx k^2$, for all $k \in N$, which implies

$$\sup_k M\left(\frac{d_\infty(\Delta_2 Y_k, \overline{0})}{\rho}\right) = \infty, \quad \text{for each fixed } \rho > 0. \quad (40)$$

Hence, $(Y_k) \notin \ell_\infty^F(M,\Delta_2)$.

Thus the classes of sequences $\ell_\infty^F(M,\Delta_m^n)$, $c^F(M,\Delta_m^n)$, and $c_0^F(M,\Delta_m^n)$ are not symmetric in general. □

Note 1. For $m = n = 0$, the class of sequences $\ell_\infty^F(M)$ and $c^F(M)$ are symmetric. For $m \le 1$ and $n \le 1$, the class of sequences $c_0^F(M,\Delta_m^n)$ is symmetric.

Proposition 6. *The classes of sequences $\ell_\infty^F(M,\Delta_m^n)$, $c^F(M, \Delta_m^n)$, $c_0^F(M,\Delta_m^n)$ are not convergence-free in general.*

Proof. The result follows from the following example.

Example 7. Let $m = 4$ and $n = 1$. Let $M(x) = x^3$, for all $x \in [0,\infty)$. Consider the sequence (X_k) defined by

$$X_k(t) = \begin{cases} 1, & \text{for } k \in N,\ t = \left(\frac{1}{k}, \frac{1}{k}, \frac{1}{k}, \ldots\right), \\ 0, & \text{otherwise.} \end{cases} \quad (41)$$

Then,

$$\Delta_4 X_k(t) = \begin{cases} 1, & \text{for } k\in N,\ t = \left(\frac{4}{k(k+4)}, \frac{4}{k(k+4)}, \frac{4}{k(k+4)}, \ldots\right), \\ 0, & \text{otherwise.} \end{cases} \quad (42)$$

Hence we have $d_\infty(\Delta_4 X_k, \overline{\theta}) = 4/k(k+4)$, which implies $(X_n) \in c_0^F(M,\Delta_4) \subset c^F(M,\Delta_4) \subset \ell_\infty^F(M,\Delta_4)$.

Consider the sequence (Y_k) defined by

$$Y_k(t) = \begin{cases} 1, & \text{for } k \in N,\ t = (k^2, k^2, k^2, \ldots), \\ 0, & \text{otherwise,} \end{cases} \quad (43)$$

so that

$$\Delta_4 Y_k(t) = \begin{cases} 1, & \text{for } k \in N, \\ & t = (-(8k+16), -(8k+16), \ldots), \\ 0, & \text{otherwise.} \end{cases} \quad (44)$$

Thus $d_\infty(\Delta_4 Y_k, \overline{\theta}) = 8k+16$, for all $k \in N$, which implies

$$\sup_k M\left(\frac{d_\infty\left(\Delta_4 Y_k, \overline{\theta}\right)}{\rho}\right) = \infty, \quad \text{for some } \rho > 0. \quad (45)$$

Thus $(Y_k) \notin \ell_\infty^F(M,\Delta_4)$.

Hence the classes of sequences $\ell_\infty^F(M,\Delta_m^n)$, $c^F(M,\Delta_m^n)$, $c_0^F(M,\Delta_m^n)$ are not convergence-free in general. □

Theorem 8. *Let M, M_1, and M_2 be Orlicz functions satisfying Δ_2-condition. Then, for $Z = \ell_\infty^F, c^F$, and c_0^F,*

(i) $Z(M_1,\Delta_m^n) \subseteq Z(M^\circ M_1,\Delta_m^n)$,

(ii) $Z(M_1,\Delta_m^n) \cap Z(M_2,\Delta_m^n) \subseteq Z(M_1+M_2,\Delta_m^n)$.

Proof. (i) Let $(X_k) \in Z(M_1,\Delta_m^n)$. Consider $\varepsilon > 0$ and $\eta > 0$ such that $\varepsilon = M(\eta)$.

Then,

$$M_1\left(\frac{d_\infty(\Delta_m^n X_k, L)}{\rho}\right) < \eta, \quad \text{for some } \rho > 0. \quad (46)$$

Let

$$Y_k = M_1\left(\frac{d_\infty(\Delta_m^n X_k, L)}{\rho}\right), \quad \text{for some } \rho > 0. \quad (47)$$

Since M is continuous and non-decreasing, we get

$$M(Y_k) = M\left(M_1\left(\frac{d_\infty(\Delta_m^n X_k, L)}{\rho}\right)\right) < M(\eta) = \varepsilon, \quad \text{for some } \rho > 0, \quad (48)$$

which implies $(X_k) \in Z(M^\circ M_1,\Delta_m^n)$.

This completes the proof.

(ii) Let $(X_k) \in Z(M_1,\Delta_m^n) \cap Z(M_2,\Delta_m^n)$.

Then,

$$M_1\left(\frac{d_\infty(\Delta_m^n X_k, L)}{\rho}\right) < \varepsilon, \quad \text{for some } \rho > 0,$$
$$M_2\left(\frac{d_\infty(\Delta_m^n X_k, L)}{\rho}\right) < \varepsilon, \quad \text{for some } \rho > 0. \quad (49)$$

The proof follows from the equality

$$\begin{aligned}(M_1+M_2)\left(\frac{d_\infty(\Delta_m^n X_k, L)}{\rho}\right) &= M_1\left(\frac{d_\infty(\Delta_m^n X_k, L)}{\rho}\right) + M_2\left(\frac{d_\infty(\Delta_m^n X_k, L)}{\rho}\right), \\ &< \varepsilon+\varepsilon = 2\varepsilon, \quad \text{for some } \rho > 0,\end{aligned} \quad (50)$$

which implies that $(X_k) \in Z(M_1+M_2,\Delta_m^n)$.

This completes the proof. □

Proposition 9. *One has $Z(M,\Delta_m^i) \subset Z(M,\Delta_m^n)$, for $0 \le i < n$, for $Z = \ell_\infty^F$, c^F, and c_0^F.*

Proof. Let $(X_k) \in \ell_\infty^F(M, \Delta_m^{n-1})$. Then we have,

$$\sup_{k\geq 1} M\left(\frac{d_\infty\left(\Delta_m^{n-1}X_k, \overline{\theta}\right)}{\rho}\right) < \infty. \tag{51}$$

Now we have

$$\begin{aligned}
&\sup_{k\geq 1} M\left(\frac{d_\infty\left(\Delta_m^{n}X_k, \overline{\theta}\right)}{\rho}\right) \\
&\quad= \sup_{k\geq 1} M\left(\frac{d_\infty\left(\Delta_m^{n-1}X_k - \Delta_m^{n-1}X_{k+m}, \overline{\theta}\right)}{\rho}\right) \\
&\quad\leq \frac{1}{2}\sup_{k\geq 1} M\left(\frac{d_\infty\left(\Delta_m^{n-1}X_k, \overline{\theta}\right)}{\rho}\right) \\
&\qquad+ \frac{1}{2}\sup_{k\geq 1} M\left(\frac{d_\infty\left(\Delta_m^{n-1}X_{k+m}, \overline{\theta}\right)}{\rho}\right) < \infty.
\end{aligned} \tag{52}$$

Proceeding in this way, we have $Z(M, \Delta_m^i) \subset Z(M, \Delta_m^n)$, for $0 \leq i < n$, for $Z = \ell_\infty^F, c^F$, and c_0^F.

This completes the proof. □

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